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Mathematical modelling in the junior secondary years:

An approach incorporating mathematical technology

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Two paper folding activities are used to demonstrate how mathematical modelling and technology can support mathematics learning.

Introduction

Mathematical models are conceptual processes that use mathematics to describe, explain, and/or predict the behaviour of complex systems. This article is written for teachers of mathematics in the junior secondary years (including out-of-field teachers of mathematics) who may be unfamiliar with mathematical modelling, to explain the steps involved in developing a model and the differences between modelling and traditional problem solving.

The Rationale for the *Australian Curriculum: Mathematics* in Years F–10 explains that the curriculum "encourages teachers to help students become self-motivated, confident learners through inquiry and active participation in challenging and engaging experiences" by teaching mathematics proficiencies that "enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently" (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2017). In the senior subjects, mathematical modelling is a feature of both General Mathematics (for example, in Unit 3) and Mathematical Methods (also Unit 3) (ACARA, 2017). Despite the lack of explicit references to mathematical modelling in the F–10 curriculum documents, we argue that problem solving should be interpreted to include modelling as an important form of mathematical inquiry and also to prepare students for what is to come in the senior years. If students are to be exposed to mathematical modelling in the junior secondary school, then teachers of junior secondary mathematics should have some understanding of the processes involved. However, empirical research suggests that "mathematics teachers might lack awareness of the significance of modelling in everyday instruction" (Siller & Kuntze, 2011, p. 33). In our experience in Australia mathematical modelling is seldom taught to younger students.

In this paper we explain the process of mathematical modelling and how it can interface with the use of mathematical technology. We then use two similar paper-folding activities to demonstrate how mathematical modelling can support the teaching of junior secondary mathematics and provide opportunities to incorporate the use of mathematical technology such as computers and graphics calculators.

Mathematical models

Models are conceptual processes used to construct, describe, explain, and/or predict the behaviour of complex systems (English, 2008). Mathematical models differ from other categories of models mainly because they focus on the structural characteristics of the systems they describe (Lesh & Harel, 2003). They are underpinned by quantitative processes (such as counting, measuring, calculating, graphing, inferring, extrapolating), although they may also include qualitative methods (such as describing and explaining).

Mathematical models can be represented using language (written and spoken), symbols, visual images/graphics (both computer- and paper-based), or experience-based metaphors. They generally have two parts: a conceptual system for describing or explaining the relevant mathematical objects and the relations between them; and procedures for generating useful outputs (Lesh & Harel, 2003). Mathematical modelling is the process of developing, evaluating, modifying, and applying mathematical models.

Mathematical modelling is also an important part of learning in subjects underpinned by mathematics, including all STEM (science, technology, engineering, and mathematics) subjects. According to English (2008), “an appreciation and understanding of the world as comprising interlocked complex systems is critical for all citizens in making effective decisions about their lives as both individuals and as community members.” (p. 5). She continues, “Modelling is not just confined to mathematics and science, however. Other disciplines including economics, information systems, finance, medicine, and the arts have also contributed in large part to the powerful mathematical models we have in place for dealing with a range of complex systems.” (p. 6).

Mathematical modelling provides learning opportunities that encourage optimal development of STEM-related skills. It is not just reaching the goal that is important, but also the interpretation of the goal, the information provided, and the possible steps to solution (English, 2008). Modelling allows students to develop useful skills such as interpreting, thinking, communicating, justifying, revising, refining, and extending that can be applied more generally (Doyle, 2006; English, 2008).

The process of mathematical modelling

The mathematical modelling process is a cycle that starts with a problem (usually grounded in reality) that requires a model to describe, explain, or predict the behaviour of a system or process. The *USA Common Core State Standards for Mathematics* (CCSSI, 2010) identifies six actions that are required to complete a modelling task, described below.

1. **Identifying the variables:** These are the elements or features relevant to the situation to be modelled (English, 2008; Stillman, Brown, & Geiger, 2015). Research and data collection may be necessary. As a system can include a large number of variables, formulating a model can be complex. Reducing the number of variables by making certain assumptions can simplify the situation without detracting from the usefulness of the model in explaining or predicting the real world.
2. **Formulating a model from the variables:** Simpler problems can be explored and data can be represented in other forms to identify relations and patterns that can be refined, extended, generalised, and abstracted back to the original problem. The generalised situation is mathematised into a useable form, for example, an equation or graph. If mathematical technology is used to assist in these processes, the model may be in the form of a computer program.

3. **Performing operations using the model:** This usually requires calculations and/or computer simulations to produce numerical outputs.
4. **Interpreting the results:** The results generated by the model are related back to the original context.
5. **Validating the mathematical model:** This is necessary because the modelling process is a trade-off between simplicity and accuracy. The challenge is to find a model where the real-world situation is simplified through the use of assumptions that remove complexity, but which nevertheless yields results that approximate reality. Models can be validated by comparing their outputs with known values. The impact of any assumptions can be tested by examining the effect of varying those assumptions; ideally, changes to the assumptions make little difference to the outputs. If the model does not yield suitable results, then it is necessary to refine the model by reviewing and adapting the variables used and assumptions made when originally developing the model (hence the reference to the modelling cycle).
6. **Reporting conclusions.**

The process is summarised in Figure 1.

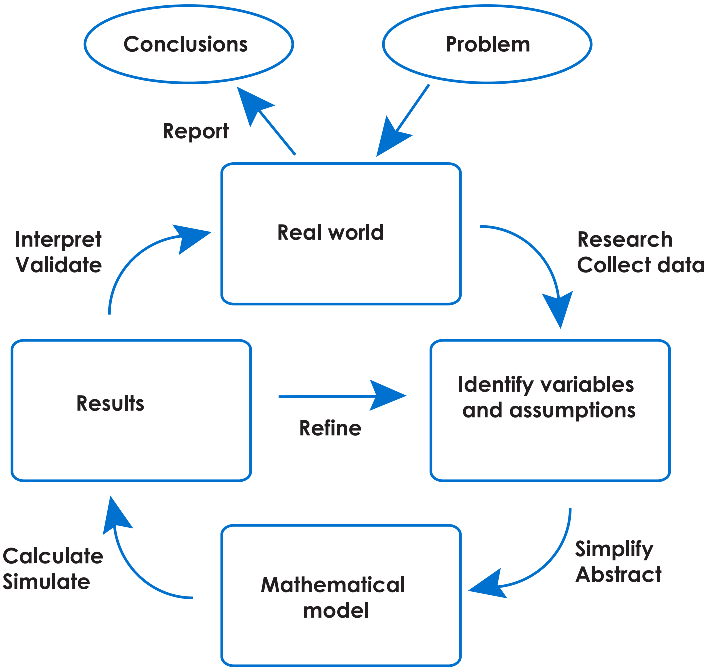


Figure 1. Mathematical modelling cycle.

More than traditional problem solving

Although often associated with problem solving, mathematical modelling involves more than traditional problem solving (Lesh & Harel, 2003). Modelling is associated with open-ended, inquiry-based investigations. Specifically, modelling differs from traditional problem solving in several respects (Doyle, 2006; English, 2008; Lesh & Harel, 2003).

- It goes beyond traditional problem solving where the key data are presented "up front" and students choose a strategy to produce a single, usually brief, response. In contrast, modelling problems embed the information within the problem context for students to discover as they work with the problem.
- The problems are often multidisciplinary, relating to learning areas beyond mathematics.

- The strategies and processes used in modelling (such as constructing, describing, explaining, predicting, and representing, together with organising, coordinating, quantifying, and transforming data) tend to differ from those used for traditional problem solving.
- The development of a model involves iterative cycles where ideas are tested, retested, and revised.
- A model can deal with more than one instance: it is reusable, shareable, and/or modifiable.
- Modelling problems are multifaceted, with the final product employing a variety of representational formats including text, graphs, tables, diagrams, spreadsheets, and oral reports.
- Students may make initial assumptions to reduce the complexity of the model, yet produce results that are useful approximations of the real world.
- It is usually clear why the model is needed, providing a way of evaluating it.
- There may be several acceptable approaches, representations, and solutions.
- The results of the model are usually predictions or estimates.

In a study of 83 tasks that were presented as mathematical modelling in some popular USA mathematics textbooks, the vast majority of the student activities involved performing operations and interpreting results (Meyer, 2015). This means that many tasks, despite being labelled as mathematical modelling, involved little more than traditional problem-solving methods. In our experience, the approach of Australian textbooks is similar, possibly due to a limited interpretation of the *Australian Curriculum* documents.

Paper boxes

We use two mathematical modelling tasks to demonstrate the richness and variety possible if mathematical modelling is used as a junior secondary learning experience. The tasks are based on the construction of an open box from a single sheet of paper, but offer very different learning experiences. Both tasks can be used to support the teaching of volume, drawing on the formula V = Ah (which is, of itself, a mathematical model). They require students to work in groups to produce many boxes of varying sizes and generate a bivariate data set that links a linear measurement to a volume. This process of collecting data is the starting point for the development of the mathematical model. The first task, which we call the 'classic box task', requires students to cut identical squares from the four corners of a sheet of rectangular paper and then fold up the edges to form an open box (as shown in Figure 2). By changing the sizes of the cut-out squares, boxes of differing volumes are created. The task is to determine the size of the cut-out square that maximises the volume of the box. Although traditionally used as an example of optimisation using calculus, this task is equally suited to providing modelling opportunities for younger students.

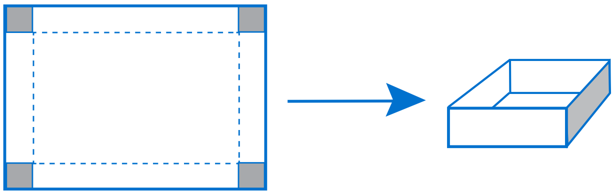


Figure 2. Classic box task.

Task two we call 'maths-in-a-box' (YuMi Deadly Centre, 2017). Students are provided with folding instructions (https://www.youtube.com/watch?v=PPif9rA1h7U) to produce an origami box from a square piece of paper. The volume of the box depends only on the length of one side of the paper square. The task is to model the relationship between that length and the volume. The complex folding process to obtain the box ensures that the relationship is not obvious to students. It requires them to develop a model based on the data that they collect after making boxes from several different-sized squares.

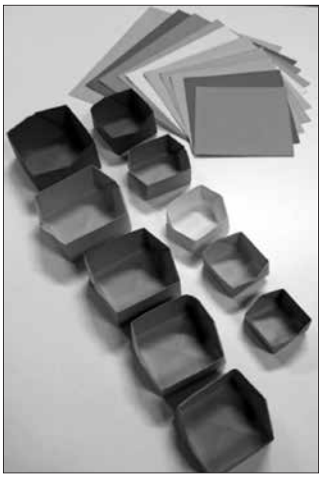


Figure 3. Maths-in-a-box task.

Both activities demonstrate the characteristics of a modelling task. They require the data to be collected then transformed into tabular, graphical, and algebraic forms so that students can describe, explain, and predict outcomes. Similarly, both tasks allow a range of possible approaches and can elicit a multifaceted response from students requiring more than a single answer. Both tasks have the potential to generate an algebraic model, which can be used to make predictions about hypothetical boxes. Some might consider these predictions to be exact, but not when considered within the context of the task, and the assumptions made about the accuracy of cuts and folds. Depending on the approach taken by students, there is scope for the models to be revised and improved.

When the teaching objectives of each task are considered, it is clear that the tasks differ. The classic box task requires the development of a relatively simple mathematical model, capable of being represented in spreadsheet, graphical or algebraic forms. This is within the ability of most Year 8 or 9 students. However, the main purpose of the task is to explore the infinite number of possible solutions, with the intention of finding the best (optimal) solution. It allows students to learn that there can be many solutions to a problem and that mathematics can be used to find the optimal solution, preparing the way for future calculus studies.

In contrast, the main objective of the maths in a box task is the development of the mathematical model—a more challenging process because of the complex folds needed to make the box. To find the size of the paper needed to produce a box with any given volume, a model is required rather than a specific solution. Many representations of the model are possible, but because the box is made by diagonally folding the square piece of paper, an algebraic solution requires a knowledge of Pythagoras' theorem and surds.

Differentiation

One advantage of mathematical modelling as a learning experience is that students are able to engage with a modelling task at a level appropriate to their mathematical development. Some students may not be able to progress beyond collecting data and showing relationships between variables graphically. However, a graph may be sufficient for them to obtain results and make predictions. Other, more capable, students may be able to represent the model algebraically and explore options for modifying, generalising, and applying the model, demonstrating a more sophisticated understanding of the relevant mathematical relations. For example, the classic box task could be extended by (a) modifying the model to accommodate different paper sizes (since the task is initially restricted to a sheet of paper of a fixed size), and (b) the real-world applications of the model. Teachers should have pre-prepared prompts to extend and enrich a mathematical modelling task to a level appropriate for each student, for example, “What if …?”, “How could we …?”, “Can you do this another way?”, and “How/where could we use this model?”

Using technology in mathematical modelling

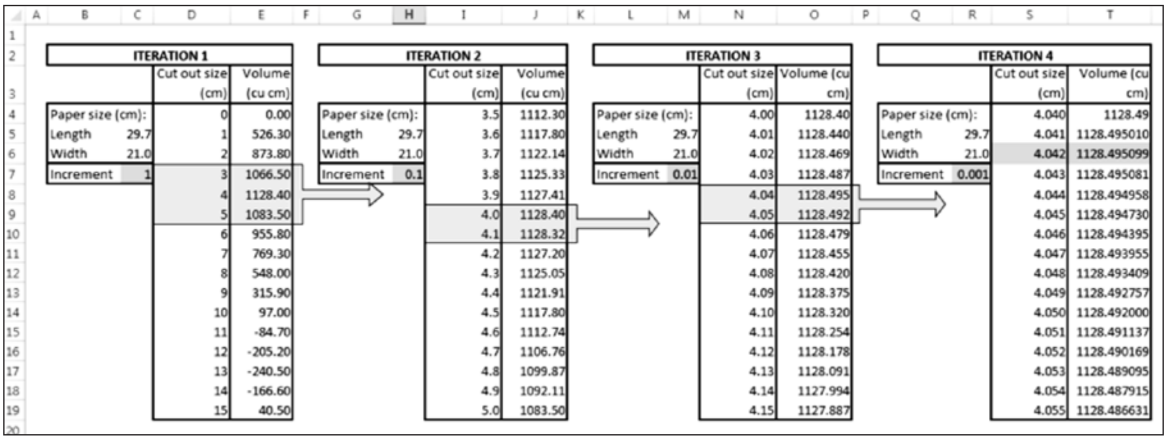
Mathematical technology, such as calculators (scientific, graphics, and computer algebra system [CAS]), spreadsheets, and computer-based graphing software, has the potential to enrich the nature of mathematical modelling tasks that can be used in the junior secondary classroom, depending on how the teacher chooses to use the technology. A number of authors have explored students’ use of technology, especially in mathematics. Galbraith, Renshaw, Goos, and Geiger (1999) identified four roles for mathematical technology: (a) technology as master, where activity is either limited to those operations over which the student has technical competence or results in uncritical acceptance of the output generated, irrespective of its accuracy; (b) technology as servant, where the technology is used as a reliable time-saving replacement for tedious manual computations; (c) technology as partner, where the technology becomes “a friend to go exploring with, rather than merely a producer of results” (p. 225) and where the student understands that the process is more than just seeking a required result; and (d) technology as extension of self, where the technology becomes an essential part of the student’s tool-kit, along with mathematical knowledge, mental processes and on-paper skills. The use of mathematical technology should be regarded as a skill to support higher order thinking.

Both paper box tasks can be enhanced through the appropriate use of technology. They both require the initial collection of data to be recorded in a table and displayed as a scatter plot—an ideal application for spreadsheet or graphing software. If students are duplicating on-paper methods using technology, perhaps with a larger data set, then it is likely that the technology is being used as a servant to do the routine, time-consuming tasks.

How the students progress beyond this stage depends on the teacher and the prompting questions they use to direct students’ modelling activities. Prompting questions are essential if the students are to reorganise the task and utilise the technology in ways that encourage exploration of the task beyond the superficial generation of a required answer. This is the essence of mathematical modelling. Without suitable questions (e.g., “What can you see from the graph?”, “What do you think the relationship is?”, “What else do you need to know?”), there is no requirement to generate a model. In this context the use of technology complements the process of mathematical modelling. It provides the tools required to simplify the development of a model (servant role) and the means to explore, expand, and refine the model (partner and extension-of-self roles).

One thing to be avoided is the ‘black-box’ use of technology (Buchberger, 1989), such as using a spreadsheet tool to fit a curve to a data set without any understanding of the underlying regression process. This is often the case when the mathematical technology is regarded as a means to an end. Students are given a scripted button-pushing recipe or a prescribed set of spreadsheet commands that might generate a suitable model, but will not add to their understanding. Students should always have a general idea of what the technology is doing, even if they do not understand the detail of the underlying algorithm. Little or no understanding of the underlying processes may lead to errors, particularly in selecting the best mathematical model and interpreting the outcomes of the model.

How then can students be encouraged to investigate further without resorting to black-box methods? It is at this point that the two paper box activities differ. In the classic box task, the scatter plots produced by graphing the data collected by the students (that is, the volume of the paper box against the size of the square cut-outs) generate parabolic curves, outside the range of functions that junior secondary students may have encountered. However, they should be able to see that there is a point at which the volume reaches a maximum, and may be able to identify a range within which this maximum volume lies and even suggest a possible maximum value. The accuracy of this estimate depends on the data collected from the box constructions. However, if students can represent the model mathematically and use a spreadsheet to calculate the volume from incremental values of the size of the cut-out corners (Figure 4), they can zoom in on the maximum value by decreasing the size of the increments. This method allows a fruitful discussions about infinitely increasing and decreasing the increments and limits. In a similar way, students could be prompted to develop a coding solution to represent the model.



ITERATION 1			
		Cut out size (cm)	Volume (cu cm)
Paper size (cm):		0	0.00
Length	29.7	1	526.30
Width	21.0	2	873.80
Increment	1	3	1066.50
		4	1128.40
		5	1083.50
		6	955.80
		7	769.30
		8	548.00
		9	315.90
		10	97.00
		11	-84.70
		12	-205.20
		13	-240.50
		14	-166.60
		15	40.50

ITERATION 2			
		Cut out size (cm)	Volume (cu cm)
Paper size (cm):		3.5	1112.30
Length	29.7	3.6	1117.80
Width	21.0	3.7	1122.14
Increment	0.1	3.8	1125.33
		3.9	1127.41
		4.0	1128.40
		4.1	1128.32
		4.2	1127.20
		4.3	1125.05
		4.4	1121.91
		4.5	1117.80
		4.6	1112.74
		4.7	1106.76
		4.8	1099.87
		4.9	1092.11
		5.0	1083.50

ITERATION 3			
		Cut out size (cm)	Volume (cu cm)
Paper size (cm):		4.00	1128.40
Length	29.7	4.01	1128.440
Width	21.0	4.02	1128.469
Increment	0.01	4.03	1128.487
		4.04	1128.495
		4.05	1128.492
		4.06	1128.479
		4.07	1128.455
		4.08	1128.420
		4.09	1128.375
		4.10	1128.320
		4.11	1128.254
		4.12	1128.178
		4.13	1128.091
		4.14	1127.994
		4.15	1127.887

ITERATION 4			
		Cut out size (cm)	Volume (cu cm)
Paper size (cm):		4.040	1128.49
Length	29.7	4.041	1128.495010
Width	21.0	4.042	1128.495099
Increment	0.001	4.043	1128.495081
		4.044	1128.494958
		4.045	1128.494730
		4.046	1128.494395
		4.047	1128.493955
		4.048	1128.493409
		4.049	1128.492757
		4.050	1128.492000
		4.051	1128.491137
		4.052	1128.490169
		4.053	1128.489095
		4.054	1128.487915
		4.055	1128.486631

Figure 4. Screen-shot of the data generated by a spreadsheet model of the classic box task.

In contrast, the maths-in-a-box activity does not provide students with an obvious link between the volume of the box and the side length of the paper square. However, they can draw on their existing knowledge to avoid the black-box problem. Students can establish that the box is a rectangular prism (without a lid), and then unfold it to see the sections that form the sides. Having collected data about boxes made from various-sized paper squares, they can plot the lengths of each of the sides (length, breadth, and height) against the paper size. These relationships are linear—within the understanding of junior secondary students. The equations of these linear relationships can be determined by students using a variety of on-paper or electronic methods.

The volumes of the various boxes can now be calculated, initially using the side lengths as inputs, but then linking the volume to the paper size. As shown in Figure 5, graphing technology allows students to examine the relationship between the volume of the box (boxvol) and paper length (plength).

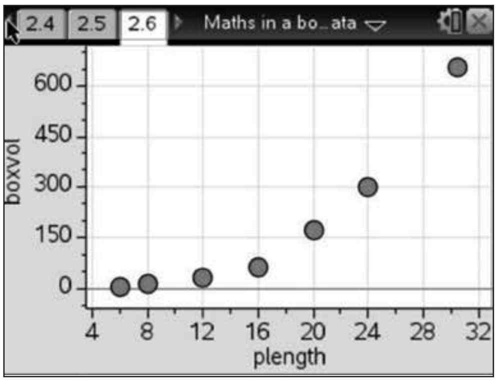


Figure 5. Scatter plot of the data generated by the maths-in-a-box task.

Substitution of the linear equations for the length, breadth, and depth of the box into the basic volume formula for a prism leads to an algebraic model linking the box volume to the paper length. Again, graphing technology provides immediate visual feedback about the quality of the model. Although the underlying function is a cubic, a family of curves that may not be familiar to junior secondary students, the black-box problem has been avoided by drawing on students' existing knowledge to develop the model. Our experiences with junior secondary students suggest that students will enthusiastically construct boxes and collect and plot data but may require some prompting to engage in the next part of the modelling process. The use of graphing software is a powerful motivator to encourage students to persist.

Discussion and conclusion

The two paper box examples demonstrate how mathematical modelling tasks can be implemented in the junior secondary classroom. The two apparently similar tasks lead in different directions, allowing the teacher to select the one that better suits the students, their level of mathematical development, and the lesson objectives. They also focus on different aspects of the modelling process: the classic box task relies on a relatively simple model and focuses on the outputs of that model, whereas the maths-in-a-box task focuses on the process of developing and evaluating the model. Other tasks could provide the model and ask students to evaluate and refine it. Modelling tasks do not need to focus on all aspects of the mathematical modelling process, provided that the open-ended nature of the tasks is not jeopardised.

The use of age-appropriate mathematical modelling tasks in the junior secondary years has several benefits. Mathematical modelling tasks:

- have clear connection to reality;
- provide an engaging, kinaesthetic approach to learning mathematics;
- encourage student creativity and independence;
- show mathematics to be a body of interconnected knowledge rather than a series of discrete topics;
- encourage students to use technology in various roles (servant, partner, and extension of self);
- can prepare for some ideas that students will encounter in senior mathematics; and
- develop the skills of modelling needed for success in senior mathematics courses.

Whilst this paper has focused on modelling in the context of mathematics lessons, mathematical modelling is relevant to all STEM subjects (and others, such as economics). If students are first taught the process of mathematical modelling in mathematics lessons, this understanding can be drawn upon later, when required, in other subjects.

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