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The most used formula in the world

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Introduction

Which is the most used formula in the world? A recent author claimed that 'it is likely that the Black and Scholes formula for calculating the values and risk characteristics of options is the most used of any mathematical formula in the world since it is used hundreds of millions of times per day as banks and financial institutions continually update and analyse their portfolios of options' (Price, 1997). The claim itself may be debatable, but there is no denying that mathematics is undergoing boom times in the financial world.

This paper aims to gives teachers some background as to where and why the Nobel Prize-winning equation of Fischer Black and Myron Scholes (and Robert Merton) arises, and how students might be given a feel for its nature. The road is fairly smooth, but we beg the reader's patience with some of the financial jargon that is necessary to describe what is going on.

Options

First of all, when someone in the markets talks about a derivative they are most likely not talking about an instantaneous rate of change. Consider the following scenarios:

- on 1 December I go to the stock exchange and buy 100 shares in BHP (now BHP Billiton) at $9.00 per share, paying $900 in all;
- on 1 December I sign a contract in which I promise to buy 100 shares in BHP at $9.50 per share on 1 January, paying nothing now;
- on 1 December I sign a contract in which I agree to pay $30 now, $0.30 per share, and which allows me to buy 100 shares in BHP at $9.50 per share on 1 January.

In the first case I have bought something in a *physical* market because I now own a part of BHP. In the other two cases I do not own a part of BHP but I have a contract that guarantees a price at which I can purchase shares in the future. The first of these is called a *futures contract*, a fixed agreement for a future transaction, while the second is called an *option*, since I have the option of not buying the shares. For example, if on 1 January BHP shares were selling for $9.00, I would simply buy the shares, losing the $30 I paid for the option but gaining on the price difference. Such an option is termed a *call option* since it allows us to buy the shares; a *put option* is a contract that allows us to sell shares at specified price. We will focus on call options.

Just like physical stocks, futures and options contracts have value and so can be traded. Their value is derived from the value of the underlying stock, in this case the price of BHP shares, and so they are known as *derivatives*.

Why are futures and options of interest? The historical use of these contracts is in *hedging* prices. Suppose you are going for an overseas holiday in a month's time and need US$100. If the exchange rate was currently 0.5000 then you could buy the currency for AU$200.00 today. But perhaps you have heard on the news that economists expect the Australian dollar to strengthen — if you waited then you might be able to get that US$100 a bit more cheaply. However, there is uncertainty in this. It will be hard to plan the rest of your spending since you will not know how much AU$ you will need for the exchange; it may even be more expensive than today! Now if someone is offering a one-month futures contract at 0.5200 then you could agree to it, guaranteeing a price of AU$192.31. You will miss any benefit if the rate goes above 0.5200, but you have stability and can better plan for the future. This is exactly the situation faced by many businesses. Such hedging provides the stability they need to plan their actions.

However, options have another interesting use. Returning to the BHP example, suppose I have bought the 100 shares for $9.00 each on 1 December and that I then sell them on 1 January when they are $10.00 per share, making a profit of $100. As a proportion of my investment, this $100 is $100/$900 = 11.1%, not too bad.

Now suppose instead I buy the option for $30 to allow me to buy 100 shares at $9.50 on 1 January. When that date arrives they are $10.00 on the market, so I can exercise the option to buy at $9.50, selling immediately at $10.00, a profit of $50 from the 100 shares. (In fact the options are dealt with in a clearing house, which means I just get paid the difference, rather than doing two physical transactions.) Deduct the $30 to buy the option and I have made $20. This does not seem as good as the $100 from the other method, but what has been my investment this time? Just the $30 needed to buy the option. The proportional return is $20/$30 = 66.7%, spectacular! In fact if I had used the $900 to buy an option on 3000 BHP shares, I would have made a profit of $900, compared to $100 from the equivalent physical investment. Wow!

If you have ever watched the share market with your class then it is simple enough to try playing with options as well. We will leave it as an exercise to see why, of course, you can run into trouble with options. Some of the glorious financial disasters of history have come from dabbling in options or related products. It is useful to make a graph of the profit (or loss) against change in share price for the physical investment and for using the equivalent option. When is an option a better investment than the physical investment, and vice versa?

To summarise, we can make transactions in a physical market, such as a stock exchange or currency exchange, or we can use futures or options. These are both known as *derivatives*, since their values are derived from the values of physical investments. The underlying investment is called, curiously enough, the *underlying*.

Option pricing

In the above example we said that the option cost $0.30 per share. But where does this value come from? Given that we made such a great profit, perhaps the option was too cheap. Once a class is happy with the idea of an option, get them to come up with some of the factors that would affect the option pricing. Futures and options prices are quoted in *The Australian Financial Review*. Students can also look at how these prices are quoted to see the factors involved, and how the prices change with these factors.

One obvious element in the price of options is the current interest rate. Interest rates exist because of the time value of money. That is, if I have $100 today then I could do something with that money, such as starting a business, so that in the future I will have more than $100. If someone wants to borrow my money, which is essentially what happens if I invest it, is they will need to pay me back more than $100 to make it worth my while. Credit risk also affects interest rates. For example, the Commonwealth government can borrow at a low rate because it is very unlikely that they will not be able to pay it back, whereas banks and individuals must offer higher rates because there is chance the customer might lose money by lending it to them. We will use the idea of the *risk-free* interest rate, which ignores such things as credit risk. This is currently around 4% in Australia.

As seen in the newspaper, the option price also depends on the *strike price* and the *time to expiry*. If the strike price for a call option is very high then it is quite likely that the option will not be exercised and thus the price can be expected to be small. On the other hand, if we want to guarantee a low price for buying shares then we would expect to pay a higher premium for it. The following table shows the *offer prices* for BHP options expiring on 20 December 2001, as at 29 November 2001:

Strike Price	$7.51	$7.99	$8.23	$8.72	$9.20	$9.68	$10.17	$10.65
Offer Price	$2.43	$1.93	$1.70	$1.21	$0.77	$0.40	$0.17	$0.07

These values were obtained from the Australian Stock Exchange web site, www.asx.com. The underlying share price on the day was $9.86. Thus, for example, a strike price of $7.51 would give a profit of $2.35 if the share price stayed the same. An option price of $2.43 loosely suggests that the market thinks the price will have gone up in a month's time, making a strike price of $7.51 more profitable.

The next table shows the effect on time to expiry, with offers again taken on BHP call option as at 29 November 2001. Each offer was based on a strike price of \$9.20.

Expiry Date	20 Dec 01	24 Jan 02	27 Mar 02	27 Jun 02
Offer Price	\$0.77	\$0.90	\$1.12	\$1.30

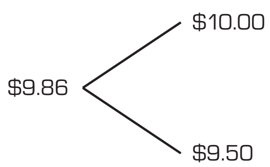
In one month's time, the share price will probably not have moved far from the current value of \$9.86 and so the value of the option is rather low. However, by June next year the share price will have had the chance to move a lot higher and so a strike price of \$9.20 could be quite valuable, and hence the higher option price.

Of course the option will depend a lot on the volatility of the underlying share price. Estimating this volatility is thus an important part of financial modelling.

Hedging

Given the vagaries of share prices, how can we hope to choose a fair price for options? The insight into solving this problem came in creating a *riskless* portfolio of shares and options. Suppose we buy Δ shares and sell one call option at a strike price of \$9.68 (so that the person who bought the option from us will be entitled to buy shares from us at the expiry date for \$9.68). The current share price is \$9.86 but we do not know the value of our portfolio since we do not know the price of the option.

However, we can imagine what could happen in a month's time, particularly if we had historical share price data to explore. For simplicities sake, suppose the share price will either go up to \$10.00 or down to \$9.50, as illustrated by the following binary tree:



If the price goes up to \$10.00 at the expiry date then the option would lose us \$0.32, since we would have to sell shares worth \$10.00 to the holder of the option for \$9.68. The Δ shares in our portfolio would be worth \$10.00Δ, giving a total value of \$10.00Δ – 0.32.

On the other hand, if the price goes down to \$9.50 then the option will be worthless and the shares will be worth \$9.50Δ. Our portfolio will be unaffected by the change in share price if

$$10.00\Delta - 0.32 = 9.50\Delta$$

Solving this gives $\Delta = 0.64$, so a riskless portfolio consists of buying 0.64 shares for each option sold. If the stock price moves to \$10.00 then this is worth 10.00(0.64) – 0.32 = \$6.08; if the price moves to \$9.50 then it is worth 9.50(0.64) = \$6.08.

Since this is a riskless portfolio it must earn the risk-free interest rate, 4%. Thus, using simple interest, the \$6.08 in one month's time is currently worth

$$\frac{6.08}{\left(1+\frac{0.04}{12}\right)} = \$6.06$$

The value of the stock price today is \$9.86, so if the current option price is c then we should have 9.86(0.64) – c = 6.06. Solving this gives an option price of c = \$0.25.

In practice, we can apply this reasoning to smaller time steps, adding extra branches to our binary tree. We can find the value of the risk-free portfolio at the expiry date and then work backwards to find the value today. The share prices at the nodes of the binary tree can be estimated from historical data.

The Black-Scholes/Merton equation, the 'most used formula in the world', simply extends this approach to continuous time, updating the riskless position at every instant. The result can be simply expressed as the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where V is the option value, S is the share price with volatility σ, and r is the risk-free interest rate. Robert Merton showed how stochastic calculus (differentiation and integration where the variable may be random) could be used with this to give a continuously changing portfolio that completely eliminated risk.

Further information about the binary tree method and the continuous-time model can be found in Hull (1998) and Wilmott (1999). *The Midas Formula*, a BBC Horizon documentary from 1999, gives an excellent overview of the history of options and the resulting use of the Black-Scholes equation. (Long Term Capital Management (LTCM), a company founded by Scholes, went bust in 1998 having derivatives with a notional value of US\$1250 billion.)

Conclusion

Mathematicians are in high demand by financial institutions that are faced with an increasingly complicated system to model. Options, with their roles as hedging tools and in speculation, come in a wide variety of forms and are a vital component of modern world markets. However, the basic ideas of options are straightforward and students can easily play with them, an extension of traditional stock market games in schools. With discrete binary trees, students can also use simple mathematics to see how they might be priced in practice. This is one area where students can have the satisfaction of understanding elements of work that earned a Nobel Prize.

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