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Author(s)	Dianne Siemon
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LAUNCHING MATHEMATICAL FUTURES: THE KEY ROLE OF MULTIPLICATIVE THINKING

DIANNE SIEMON
RMIT University
dianne.siemon@rmit.edu.au

Access to multiplicative thinking has been identified as the single, most important reason for the eight-year range in mathematics achievement in Years 5 to 9. While elements of multiplicative thinking are variously represented in the Australian Curriculum, the connections between these and how they contribute to the development of multiplicative thinking over time is not entirely clear. Two aspects of Hanna Neumann's internationally respected reputation as a mathematician, teacher, researcher, and mentor will be used to frame this presentation. The first is her commitment to making the abstract accessible. The second is her passionate interest in reforming school mathematics curricula. Examples will be used to demonstrate how the abstract might be rendered accessible in the context of school mathematics and, conversely, how abstracting the everyday can help challenge long-held beliefs about learning mathematics. But the major part of this presentation will be concerned with the critical importance of multiplicative thinking in launching mathematical futures and its representation in the Australian Curriculum.

Introduction

It is an honour to have been asked to do the Hanna Neumann lecture at this, the 24th Biennial Conference of the Australian Association of Mathematics Teachers (AAMT). Hanna and Bernard Neumann came to Australia in 1963 to take up positions at the Australian National University—Hanna as a professorial fellow until her appointment to the chair of pure mathematics in 1964, and Bernard as Professor of Mathematics in the newly formed research Department of Mathematics at ANU. Convinced that mathematics education in Australia was 'lagging behind the rest of the world to a frightening extent" (Fowler, 2000), Hanna became actively involved in the Canberra Mathematical Association providing courses for secondary mathematics teachers and contributing to the discussions on the new senior secondary mathematics syllabuses in NSW.

Previous lectures have been given by those who knew Hanna personally or at least heard her speak, for example, Dr Susie Groves and Dr Peter Taylor both of whom could claim 0 degrees of separation. I can claim 1 degree of separation having undertaken my Honours year in Pure Mathematics at Monash with Colin Fox and Steve Pride both of whom went on to ANU to study under either Hanna or Bernard and complete PhDs in

Mathematics. Bernard was a capable musician and Colin—now a broadcaster for ABC Classic FM—remembers fondly his many visits to the Neumann household for musical get togethers where Hanna, always working, would join them at the end of the evening for coffee.

We are indebted to Hanna for the very many legacies she has left behind but as a mathematics educator there are two that I would like to take up in this address—the first is her deep commitment to making the abstract accessible. The second was her passion for reforming school mathematics curricula (Fowler, 2012; Newman & Wall, 1974).

A commitment to making the abstract accessible

According to Newman and Wall (1974), Hanna Neumann developed a style of teaching that made the "acquisition of very abstract ideas accessible through judicious use of more concrete examples and well-graded exercises" (p. 4). She regularly offered lectures on topics that were not considered formally in University courses but served to convey her own joy in mathematics, participating in the model-building group and introducing undergraduate students to 'new mathematics' in creative and innovative ways. She also took an active interest in the professional development of secondary teachers of mathematics. For example, in 1971 she addressed a regional meeting of teachers at Wodonga Technical School on 'Modern Mathematics—Symbolism and its importance at the secondary and tertiary levels'. An issue, many would agree, we are still grappling with today.

Having completed my undergraduate degree in pure mathematics at Monash, I had no idea at the time just how 'modern' the mathematics courses were at Monash. All I know is that when we arrived as naïve first year students having done reasonably well under the 'old mathematics curriculum', we were deep-ended into the 'new mathematics'. My first semester was a blur—nothing looked or felt like anything we had done before. We were introduced to sets, fields, and groups and were required to use a very different type of mathematical language—but by October or so, it all magically fell into place and I remember being carried away by the sheer beauty and connectedness of it all—Hanna would have been proud.

As a teacher of secondary mathematics and fledgling mathematics educator I was again lucky to be in the right place at the right time. The Study Group for Mathematics Learning (SGML) was set up by an enthusiastic group of mathematics teachers[1] who were keen to apply the 'new mathematics' in schools. Zoltan Deines spent some time in Melbourne around this time and the SGML workshops familiarised us with the use of structured materials to support a different approach to mathematics teaching and learning (e.g., Multibase Arithmetic Blocks, Attribute Blocks, Logic Tracks and games involving multiple representations and embodiments).

Exploring group theory

Some years later, encouraged by what I had learned, I incorporated my own version of Dienes' 'games' (e.g., Dienes, 1960) to illustrate the properties of mathematical groups. I was the Year 9 and 10 Coordinator at Mater Christi College in Belgrave and we were

1 Notably Vic Ryle and Ken Clements

keen to see if we could encourage more girls to continue with mathematics into Year 11 and 12, I set up an elective called ‘Advanced Maths’ which was open to all and designed to explore mathematics that was not in the curriculum through games and activities. We worked with Boolean Algebra using logic tracks, vector mappings using a rectangular courtyard, and I designed a line-dancing type of routine based on Dienes’ games to tease out the properties of cyclic groups. Each set of four girls formed a square ABCD and the moves were limited to:

N: no move, stay in the same place.
R: move one place right (A to B to C to D to A)
L: move one place left (A to D to C to B to A)
C: move to the diagonally opposite place (A to C to A, B to D to B)

Table 1. Table of moves.

	N	**R**	**L**	**C**
N	N	R	L	C
R	R	C	N	L
L	L	N	C	R
C	C	L	R	N

We also explored Modulo Arithmetic, otherwise known as clock arithmetic, in this context, specifically, {0, 1, 2, 3; + mod 4} where, for example, 2 + 3 ~ 1 mod 4, and 2 x 3 ~ 2 mod 4. This resulted in a similar table to the one above and facilitated a discussion about patterns and commonalities. In particular that each combination of moves or combination of digits under addition generated another element of the set (closed property), that for every element of each set there was another element that when combined/added resulted in the original element (identify element), that for each element there was another element that when combined with the original element resulted in the identity element (inverse), and that order of combination or addition did not matter, that is the operation in each case was commutative. This lead us to consider other properties and eventually a better, deeper understanding of the real numbers.

The case of 9C

In my second year at Mater Christi I found myself teaching a Year 9 class of girls who were intent on leaving school as early as possible (the ‘C’ stood for ‘commercial’). Financial mathematics was a core component of their ‘modified’ program but the available text treated these topics in a particularly procedural way. I decided to try a different approach. I asked the students to pinch the pages of the respective chapters between their forefinger and thumb (it amounted to about half a centimetre) and said, ‘I’m going to let you in on a secret. All of the problems in these pages are of the type n% of $m = p$’. Over the course of two weeks we selected and solved problems according to type, that is, (i) n and m known, (ii) n and p known, or (iii) m and p known, and with little regard for context (i.e., profit and loss, simple interest, discount, etc.). This might seem counterintuitive, but it worked. At the end of the two weeks not only were they happy to sit the test, they all passed with flying colours and asked if we could do more maths like that—I obliged and we used this technique to explore Pythagoras’ Theorem, which was not in their course, but it demonstrated to them that they could perform equally as well as the girls in 9P (P for 'professional'). This taught me a valuable lesson about the clarifying power of mathematical structure, and that 'real-world' contexts can sometimes get in the way of learning mathematics.

Making the everyday abstract

Sometimes, making the everyday abstract can be a useful strategy to focus on the learning involved. For example, many years ago in an effort to convince parents that rote learning the multiplication 'tables' was ultimately counter-productive, I developed a set of * tables, where * was an operator defined as follows: $a * b = ((a + b) \times (a \times b))/(b - a)$. Exerts from two * tables were provided (see Table 2) and all but a small group of parents were asked to learn these by whatever means they chose in preparation for a test in 10 minutes time. The small group were taught the meaning of the operator in terms of the rule: 'sum multiplied by product, divided by the difference reversed' and encouraged to practice applying the rule to any fact in the one and two * tables including the related facts (e.g., 4 * 1 as well as 1 * 4).

*Table 2. List of * facts.*

0 * 1 = 0	0 * 2 = 0
1 * 1 = undefined	1 * 2 = 6
2 * 1 = –6	2 * 2 = undefined
3 * 1 = –6	3 * 2 = –30
4 * 1 = –6.666...	4 * 2 = –24
5 * 1 = –7.5	5 * 2 = –23.333...
6 * 1 = –8.4	6 * 2 = –24

The test involved five 'facts', 3 * 1, 4 * 2, 6 * 1, 1 * 2 and 2 * 3. Not surprisingly, most parents remembered at least three of the facts and the best anyone, not involved in the small group did, was four out of five. Some assumed * was commutative, others complained that it was not fair as 2 * 3 was not in the list provided. By contrast, all of the small group were able to achieve five out of five correct. This prompted an extremely robust and valuable discussion about the importance of understanding the operator involved and not just relying on memory however effective this was in the short term.

Another example of the benefits of making the everyday abstract arose from inviting primary pre-service teachers to construct a completely new set of names and symbols for the digits 0 to 9 then brainstorm what might be involved in teaching these to a group of five-year-olds. This also led to a robust discussion on the nature of mathematics learning and our assumptions as teachers. I have not tried it, but extending this activity to another number base and generating multi digit numbers might also be worthwhile.

Reforming mathematics curriculum

In 1964 Hanna Neumann was actively engaged in the discussions on the new, senior secondary mathematics syllabuses in NSW and, as Newman and Wells (1974) point out, “[it] was undoubtedly her work in evaluation of the draft proposals and her energetic work on suggestions for improvements, which earned for the Canberra Mathematical Association a reputation for trenchant and constructive criticism” (p. 6). It is in this same spirit that I offer the following commentary on mathematics curriculum in general and the role and place of multiplicative thinking in particular.

The crowded curriculum and the lack of succinct, unambiguous guidelines about the key ideas and strategies needed to make progress in school mathematic have been the concern of teachers of mathematics for many years. This is particularly the case for Number which successive mathematics curricula and text books have tended to represent as long lists of disconnected ‘topics’ that value the reproduction of relatively low-order skills and competencies rather than promoting deep understanding of key ideas, generalisation and problem solving (Siemon, 2011a).

While the importance of focussing on ‘big ideas’ is widely recognised (e.g., Charles, 2005), there is little agreement about what these are or how these are best represented to support the teaching and learning of mathematics. For example, what might be a ‘big idea’ from a purely mathematical perspective (e.g., set theory), may not be a ‘big idea’ from a pedagogical perspective. That is, ‘big ideas’ need to be both mathematically important and pedagogically appropriate to serve as underlying structures on which further mathematical understanding and confidence can be built (Siemon, Bleckly & Neal, 2012). The *Curriculum Focal Points for Pre-Kindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM, 2006) go some way towards achieving this goal by providing a more detailed account of ‘important mathematics’ at each grade level for K to 8. But big ideas are notoriously difficult to accommodate in curriculum documents as Hanna Neumann experienced in her endeavours to introduce the big ideas of ‘modern mathematics’ into the NSW mathematics curriculum in the sixties. But this does not mean we should not engage with this slippery notion. Big ideas serve a useful purpose in that they operate as a test of curriculum coherence and serve as interpretive lenses through which skill-based content descriptors can be examined in more depth.

Big ideas in mathematics

For the purposes of the *Assessment for Common Misunderstandings* (Department of Education and Early Childhood Development, 2007; Siemon, 2006) and the *Developmental Maps* (Siemon, 2011b), which were developed for the Victorian Department of Education and Early Childhood Development, a ‘big idea’ in mathematics:

- is an idea, strategy, or way of thinking about some key aspect of mathematics without which, students’ progress in mathematics will be seriously impacted;
- encompasses and connects many other ideas and strategies;
- serves as an idealised cognitive model (Lakoff, 1987), that is, it provides an organising structure or a frame of reference that supports further learning and generalizations;
- cannot be clearly defined but can be observed in activity (Siemon, 2006, 2011b).

The big ideas identified for this purpose are shown in Table 3. The rationale for the choice of number and for considering multiplicative thinking in particular will be addressed in more detail below.

Table 3. Big Ideas identified for the Assessment for Common Misunderstanding *Tools*

By the end of	'Big Idea'
Foundation	*Trusting the Count*—developing flexible mental objects for the numbers 0 to 10 (2 tools)
Year 2	*Place-value*—the importance of moving beyond counting by ones, the structure of the base 10 numeration system (4 tools)
Year 4	*Multiplicative thinking*—the key to understanding rational number and developing efficient mental and written computation strategies in later years (6 tools)
Year 6	*Partitioning*—the missing link in building common fraction and decimal knowledge and confidence (7 tools)
Year 8	*Proportional reasoning*—extending what is known about multiplication and division beyond rule-based procedures to solve problems involving fractions, decimals, per cent, ratio, rate and proportion (8 tools)
Year 10	*Generalising*—skills and strategies to support equivalence, recognition of number properties and patterns, and the use of algebraic text without which it is impossible to engage with broader curricula expectations at this level (4 tools)

Multiplicative thinking

The capacity to think multiplicatively is crucial to success in further school mathematics. It underpins nearly all of the topics considered in the middle years and beyond, and lack of it or otherwise is the single most important reason for the eight-year range in mathematics achievement in Years 5 to 9 (Siemon, Virgona & Corneille, 2001). Hence the choice of number for the 'big ideas' listed above.

Multiplicative thinking involves recognising and working with relationships between quantities. In particular, it supports efficient solutions to more difficult problems involving multiplication and division, fractions, decimal fractions, ratio, rates and percentage. Although some aspects of multiplicative thinking are available to young children, multiplicative thinking is substantially more complex than additive thinking and may take many years to achieve (Vergnaud, 1983; Lamon, 2007). This is because multiplicative thinking is concerned with processes such as replicating, shrinking, enlarging, and exponentiating that are fundamentally more complex, rather than the more obvious processes of aggregation and disaggregation associated with additive thinking and the use of whole numbers (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2011).

The *Scaffolding Numeracy in the Middle Years* (SNMY) research project (see Siemon, Breed, Dole, Izard & Virgona, 2006) was designed to explore the development of multiplicative thinking in Years 4 to 8. Multiplicative thinking was seen to be characterised by:

- a capacity to work flexibly and efficiently with an extended range of numbers (i.e., larger whole numbers, decimals, common fractions, ratio and per cent),
- an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion, and
- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms).

The SNMY project used rich tasks in a pen and paper format to test a hypothetical learning trajectory for multiplicative thinking in Grades 4–8 (Siemon et al., 2006). Item response theory (e.g., Bond & Fox, 2001) was used to identify eight qualitatively different categories of responses, which subsequently lead to a *Learning and Assessment Framework for Multiplicative Thinking* (LAF) comprised of eight 'zones' representing increasingly sophisticated levels of understanding (see Table 4). Rich descriptions were developed for each zone and teaching advice was provided in the form of what needed to be *consolidated and established* and what needed to be *introduced and developed* to scaffold multiplicative thinking to the next zone.

Table 4. The Learning Assessment Framework for Multiplicative Thinking (Siemon et al., 2006)

Zone 1: Solves simple multiplication and division problems involving relatively small whole numbers but tends to rely on drawing, models and count-all strategies. May use skip counting for groups less than 5. Makes simple observations from data and extends simple number patterns. Multiplicative thinking (MT) not really apparent as no indication that groups are perceived as composite units, dealt with systematically, or that the number of groups can be manipulated to support more efficient calculation
Zone 2: Counts large collections efficiently—keeps track of count but needs to see all groups. Shares collections equally. Recognises small numbers as composite units (e.g., can count equal groups, skip count by twos, threes and fives). Recognises multiplication needed but tends not to be able to follow this through to solution. Lists some of the options in simple Cartesian product situations. Some evidence of MT as equal groups/shares seen as entities that can be counted.
Zone 3: Demonstrates intuitive sense of proportion. Works with useful numbers such as 2 and 5 and intuitive strategies to count/compare groups (e.g., doubling, or repeated halving to compare simple fractions). May list all options in a simple Cartesian product, but cannot explain or justify solutions. Beginning to work with larger whole numbers and patterns but tends to rely on count all methods or additive thinking (AT).
Zone 4: Solves simple multiplication and division problems involving two-digit numbers. Tends to rely on AT, drawings and/or informal strategies to tackle problems involving larger numbers, decimals and/or less familiar situations. Tends not to explain thinking or indicate working. Partitions given number or quantity into equal parts and describes part formally. Beginning to work with simple proportion.
Zone 5: Solves whole number proportion and array problems systematically. Solves simple, 2-step problems using a recognised rule/relationship but finds this difficult for larger numbers. Determines all options in Cartesian product situations involving relatively small numbers, but tends to do this additively. Beginning to work with decimal numbers and percent. Some evidence MT being used to support partitioning. Beginning to approach a broader range of multiplicative situations more systematically
Zone 6: Systematically lists/determines the number of options in Cartesian product situation. Solves a broader range of multiplication and division problems involving 2-digit numbers, patterns and/or proportion but may not be able to explain or justify solution strategy. Renames and compares fractions in the halving family, uses partitioning strategies to locate simple fractions. Developing sense of proportion, but unable to explain or justify thinking. Developing capacity to work mentally with multiplication and division facts
Zone 7: Solves and explains one-step problems involving multiplication and division with whole numbers using informal strategies and/or formal recording. Solves and explains solutions to problems involving simple patterns, percent and proportion. May not be able to show working and/or explain strategies for situations involving larger numbers or less familiar problems. Constructs/locates fractions using efficient partitioning strategies. Beginning to make connections between problems and solution strategies and how to communicate this mathematically
Zone 8: Uses appropriate representations, language and symbols to solve and justify a wide range of problems involving unfamiliar multiplicative situations, fractions and decimals. Can justify partitioning, and formally describe patterns in terms of general rules. Beginning to work more systematically with complex, open-ended problems.

What the data that underpins this research-based framework shows is that the transition from additive to multiplicative thinking is nowhere near as smooth or as straightforward as most curriculum documents seem to imply, and that access to multiplicative thinking as it is described here represents a real and persistent barrier to many students' mathematical progress in the middle years of schooling (Siemon & Breed, 2005; Siemon et al., 2006).

To become confident multiplicative thinkers, children need a well-developed sense of number (based on trusting the count, place-value and partitioning) and a deep understanding of the many different contexts in which multiplication and division can arise (e.g., sharing, equal groups, arrays, regions, rates, ratio and the Cartesian product). The transition from additive strategies to meaningful, mental strategies that support multiplicative reasoning more generally requires a significant shift in thinking from a count of *equal groups* and a reliance on repeated addition, to the *for each* and *times as many* ideas for multiplication that underpin all further work with multiplication, division and rational number. While the *array* and *region* ideas for multiplication can be used to support a count of equal groups, their power lies in the fact that they can be used to underpin this important shift in thinking and, ultimately, the *factor–factor–product* idea that supports the inherently multiplicative operations of equipartitioning, replicating, enlarging, shrinking and a more generalised understanding of the relationship between multiplication and division. In addition, the *region* and *for each* ideas for multiplication are also critically important in the interpretation and construction of fraction representations (for a much more detailed discussion of these ideas see Siemon, Beswick, Brady, Clark, Farragher & Warren, 2011).

The difference between additive and multiplicative thinking

The essential difference between additive and multiplicative thinking relates to the nature of the units under consideration. For addition and subtraction, "all the number meanings ... are directly related to set size and to the actions of joining or separating objects and sets" (Nunes & Bryant, 1996, p. 144). In these situations it is possible to work with the numbers involved as collections that can be aggregated or disaggregated and renamed as needed to facilitate computation.

While it is possible to use repeated addition to solve multiplication problems and repeated subtraction to solve division problems, these are essentially additive processes—the only difference is that the sets being added or subtracted are the same size. Multiplicative thinking involves much more than this and "it would be wrong to treat multiplication as just another, rather complicated, form of addition, or division as just another form of subtraction" (Nunes & Bryant, 1996, p. 144). For example, consider the following Year 4 responses to the problem: how many muffins could be made with 6 cups of milk if 2/3 cup of milk produced 12 muffins?

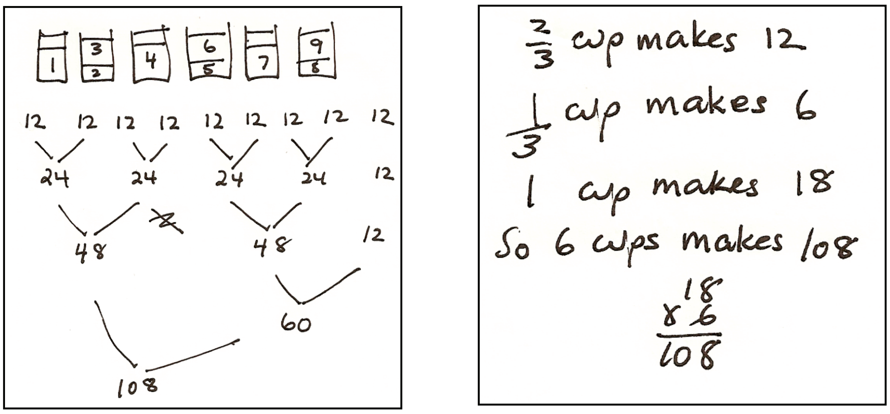

Figure 1. Two solutions to the Muffin problem (Siemon et al., 2011).

The first response uses repeated addition to determine how many two-third cups are in 6 cups, and then counting to find the total number of muffins. The second solution recognises the proportional relationship between the quantity of milk and the number of muffins. Both strategies produce the correct answer, but the first is additive whereas the second is multiplicative.

For multiplication, it is necessary to simultaneously recognise and coordinate the number of groups (multiplier) and the number in each group (multiplicand) (Anghileri, 1989; Jacob & Willis, 2001; Nunes & Bryant, 1996; Vergnaud, 1983). According to Steffe (1992), for a 'situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one composite unit is distributed over the elements of the other composite unit' (p. 264), resulting in a composite unit of composite units (e.g., see Figure 2).

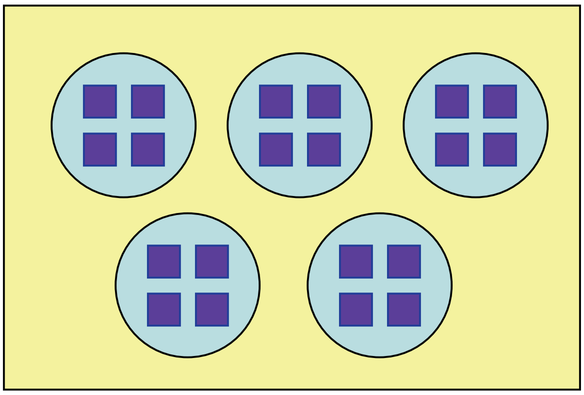

Figure 2. A composite unit of composite units.

Recognising and working with composite units introduces the distinction between *how many* (the count of composite units) and *how much* (the magnitude of each composite unit). This distinction is often overlooked in the rush to symbolise, with the result that many children interpret *3 groups of four* as successive counts of four ones rather than *3 fours* which emphasises the distribution of one composite unit over another. This has important implications for the development of multiplicative thinking and children's capacity to understand fractions. By distinguishing between the count and the unit, children are more likely to recognise the multiplicative nature of our number systems For example, the digits in the numeral 34 are both counting numbers (i.e., how many numbers), but their location or place determines the unit (i.e., how much). In the fraction ¾ the numerator indicates how many but the denominator indicates how much. They are also more likely to recognise the relative magnitude of different units (e.g., that 3 quarters is larger than 3 eighths) and the inverse relationship between *how many* and *how much* (e.g., the larger the number of shares/equal parts, the smaller each share/part).

Multiple pathways to multiplicative thinking

Nearly all of the research-based developmental frameworks for multiplication are framed in terms of counting-based strategies that ultimately terminate with a reference to the use of number fact knowledge (e.g., Department of Education & Early Childhood Development, 2010; Department of Education & Training, 2007; van den Heuvel-Panhuizen, 2001). This is not surprising given the almost exclusive focus on *equal groups* and repeated addition in the early years. However, an increasing number of researchers (e.g., Confrey, Maloney, Nguyen, Mojica & Myers, 2009; Downton, 2008; Nunes & Bryant, 1996; Schmittau & Morris, 2004) suggest that there is a parallel path to the development of multiplicative thinking based on young children's capacity to share equally and work with one-to-many relationships. For example, having explored the 'Baa-Baa Black Sheep' rhyme in literacy, a teacher posed the following question to her class of 5 and 6 year olds: 'I wonder how many bags of wool would there be if there were 5 sheep?' While most decided that there would be 15 bags of wool, what was interesting was the number of children who constructed abstract representations, in particular, representations that connected each sheep with three bags of wool (e.g., see Figure 3).

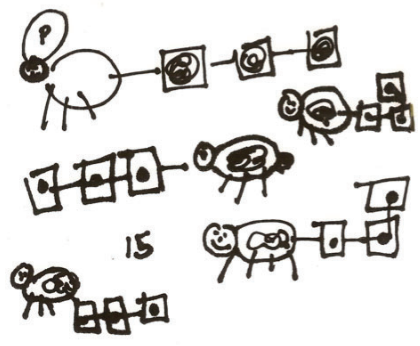

Figure 3. Five year olds solution to the Baa-Baa Black Sheep problem (Siemon et al., 2011).

This suggests that the children understood the situation in terms of *for each* sheep there are 3 bags of wool. This is essentially a ratio or times as many idea (e.g., 3 times as many bags of wool as sheep) and is quite distinct from the *equal groups* idea, even though the children invariably counted by ones to arrive at the solution of 15 bags of wool altogether.

The representation of multiplicative thinking in the Australian Mathematics Curriculum

In most English-speaking countries, multiplication and division are introduced separately, with multiplication typically considered before division. Given what is known about multiplicative thinking, young children's experience with sharing, and evidence to suggest that simple proportion problems can be solved earlier than generally expected (e.g., Confrey et al., 2009; Nunes & Bryant, 1996; Schmittau & Morris, 2004; Siemon et al., 2006), the introduction of the *Australian Curriculum: Mathematics* [ACM] provided an opportune time to reconsider when and how we introduce these important ideas. How well has it fared?

A detailed analysis of the ACMs potential for developing multiplicative thinking is included in Appendix A. Here, I shall draw on some of the observations made previously (see Siemon, Blecky &Neal, 2012) in relation to the presence or otherwise of the key ideas and strategies mentioned above—the codes in brackets refer to the content descriptors in the ACM.

While *sharing* is mentioned in Foundations (ACMNA004) the only other reference to any of the key ideas discussed above is in Year 2 where students are expected to recognise and represent "multiplication as repeated addition, groups and arrays" (ACMNA031) and "division as grouping into equal sets" (ACMNA032). This reference to division is ambiguous as it could refer to *quotition* division (where the divisor refers to size of group) or *partition* division (where the divisor refers to the number of equal groups). However, grouping a collection into equal sets and working with arrays is no guarantee of multiplicative thinking unless the focus of attention is shifted from a count of groups of the same size (additive) to a given number of groups of any size (Siemon et al., 2011). Importantly, the *region* idea is not mentioned at all and yet this underpins the *area* (*by* or *factor*) idea of multiplication (i.e., each part multiplied by every other part) which is needed to support the multiplication of larger whole numbers (e.g., 2-digit by 2-digit multiplication), the interpretation of fraction diagrams (e.g., thirds by fifths are fifteenths), and, ultimately, the multiplication and division of fractions and linear factors.

In Year 3 students are expected to "recall multiplication facts of two, three, five and ten and related division facts" (ACMNA056). This wording together with the previous (AMN026) and subsequent (AMN074) references to number sequences implies that the multiplication facts are learnt in sequence (e.g., 1 three, 2 threes, 3 threes, 4 threes, 5 threes, etc.) rather than on the basis of number of groups irrespective of size (e.g., 3 of anything is double the group and one more group).

Factors and multiples are referred to in Year 5 (ACMNA098) and Year 6 (ACMNA122), indices in Years 7 and 8 (ACMNA149 & ACMNA182), and solving problems involving specified numbers and operations across year levels (e.g., ACMNA100, ACMNA101 and ACMNA103). However, there is no suggestion of the connections between them or that something other than a repeated addition model of multiplication is needed to support a deep understanding of factors and indices (Confrey et al., 2009).

In the early years, the ACM refers to the capacity to "recognise and describe half as one of two equal pieces" (ACMNA016) and "to recognise and interpret common uses of halves, quarters and eighths of shapes and collections" (ACMNA033) but no mention is made of the important link to sharing which provides a powerful basis for the creation of equal parts and the link between fractions and partitive division (Nunes & Bryant, 1996). In Year 3, students are expected to be able to "model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole"

(ACMNA058). This suggests that fraction symbols are expected at this stage, which is problematic given the well-known difficulties associated with interpreting fraction symbols and representations (e.g., Lamon, 1999). Also, the reference to counting fractions at Year 4 (ACMNA078) could lead to an over-reliance on additive, whole number-based approaches to locating fractions on a number line at the expense of multiplicative approaches such as equipartitioning (Confrey et al., 2009; Lamon 1999).

The ACM does not refer to proportional reasoning explicitly until Year 9 where reference is made to solving problems involving direct proportion and simple rates (ACMNA208) and enlargements, similarity, ratios and scale factors in relation to geometrical reasoning (ACMMG220 & ACMMG221). While many of the prerequisite skills are included in Years 6 to 8, these appear in the form of disconnected skills. For example, "find a simple fraction of a quantity" (ACMNA127) at Year 6, "express one quantity as a fraction of another", "find percentages of quantities and express one quantity as a percentage of another" (ACMNA 155 & ACMNA158) at Year 7, and solve a range of problems involving percentages, rates and ratios (ACMNA187 & ACMNA188) at Year 8. Importantly, there is nothing to suggest how these skills relate to one another or their rich connections to multiplicative thinking more generally.

As the above discussion and the analysis in the Appendix shows, the content descriptors of the ACM have the potential to support the development of multiplicative thinking. But the extent to which this potential is realised is heavily dependent on how the descriptors are interpreted, represented, considered and connected in practice. Content descriptors do need to be in a form that is clearly assessable but, if these are taught and assessed in isolation with little attention to student's prior knowledge and the underpinning ideas and strategies, there is a substantial risk that access to multiplicative thinking will continue to elude many. On the other hand, if the content descriptors are taught and assessed in conjunction with the proficiencies, that is, conceptual understanding, procedural fluency, mathematical reasoning and mathematical problem solving, the chances of increasing access to multiplicative thinking in the middle years can be greatly enhanced.

Conclusion

Hanna Neumann left a valuable legacy to mathematics and mathematics education both here and abroad. As one of the founding members of AAMT, it is fitting that we acknowledge her contributions to school mathematics in the biennial lecture that bears her name. Her commitment to making the abstract accessible and her passion for reforming school mathematics curriculum framed this presentation. In demonstrating how group theory might be explored in the context of dance and clock arithmetic and what can be gained from working with mathematical structures, I hope you too might be prompted to consider how you might make the abstract accessible and the everyday abstract. My comments on the place of multiplicative thinking in the *Australian Curriculum: Mathematics*, are offered in the same spirit and with the same motivation that Hanna offered her suggestions and feedback on the NSW senior secondary mathematics syllabuses in the sixties—that is, the need to recognise and focus on the 'big ideas' in mathematics so that all learners have the opportunity to experience the joy of doing mathematics and to access the future that it affords.

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Appendix: Opportunities for multiplicative thinking in the ACM

Year Level	Selected ACM Content Descriptors (ACARA, 2013)	Relationship to Multiplicative Thinking
F	Subitise small collections of objects (ACMNA003)	Helps establish the notion of composite units—children 'see' a collection of 4 ones as 'four' without having to count
F	Represent practical situations to model addition and sharing (ACMNA004)	Sharing helps establish notion of equal shares, equal groups and relationship between the number of shares (how many) and the number in each share (how much)
1	Develop confidence with number sequences to and from 100 by ones from any starting point. Skip count by twos, fives and tens starting from zero (ACMNA012)	Skip counting, while essentially additive, if used as a strategy for physically counting large collections, helps establish one-many relationships and notion of composite units. Risk: limited to number naming sequences
1	Recognise, model, read, write and order numbers to at least 100. Locate these numbers on a number line (ACMNA013)	Locating numbers on a number line—if open—invites the use of multiplicative or equipartitioning based on benchmarks (e.g., it's about half)
1	Recognise and describe one-half as one of two equal parts of a whole. (ACMNA016)	Introduces multiplicative partitioning and halving. Risk is that parts will not be seen in relation to the whole
1	Investigate and describe number patterns formed by skip counting and patterns with objects (ACMNA018)	Potential to support notion of composite units. Risk is that this will be limited to additive or repeating patterns rather than multiplicative or growing patterns
1	No further reference to sharing	
2	Investigate number sequences, initially those increasing and decreasing by twos, threes, fives and ten from any starting point, then moving to other sequences. (ACMNA026)	Suggests a count of twos, threes, etc. Risk: limited to number naming sequences, preferences a count of groups as basis for multiplication facts
2	Recognise and represent multiplication as repeated addition, groups and arrays (ACMNA031)	Key representations. Risk: interpretation limited to *equal groups*, count of groups
2	Recognise and represent division as grouping into equal sets and solve simple problems using these representations (ACMNA032)	Inclusive of both forms of division (*quotition* and *partition*) . Risk: limited to count of groups, sharing not generalised to 'think of multiplication"
2	Recognise and interpret common uses of halves, quarters and eighths of shapes and collections	Potential to engage students in equipartitioning, Risk: Fraction names seen as labels for parts rather than relationships. No involvement in equipartitioning, partitioning strategies, teaching may not deal with core generalisations
2	Place value appears to be treated additively	
3	Recall multiplication facts of two, three, five and ten and related division facts (ACMNA056)	Implies memorisation of facts, unclear as to how these are represented (e.g., counts of 2 or 2 of anything). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
3	Represent and solve problems involving	Potentially supportive of multiplicative thinking if

	multiplication using efficient mental and written strategies and appropriate digital technologies	strategies based on *arrays* and *regions* and shift of thinking from size of group (count of equal groups) to number of groups (*factor* idea)
	Model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole (ACMNA058)	Strongly multiplicative where students engaged in equipartitioning strategies to construct their own fraction models and representations. Risk: focus on the name of parts not the relationship to the whole
	No reference to representations of multiplication and division	
4	Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075)	Implies memorisation of facts, unclear as to how these are represented (see above). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
	Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076)	Potentially supportive of multiplicative thinking if strategies based on shift of thinking from size of group (count of equal groups) to number of groups (factor idea), use of distributive law , etc. Risk: Strategies based on repeated addition, count all groups
	Investigate equivalent fractions used in contexts (ACMNA077)	Highly multiplicative if explored via equipartitioning strategies (e.g., halving, thirding and fifthing) and linked to *region* idea (e.g., thirds by fourths are twelfths). Risk: treated as rule-based procedure
	Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line (ACMNA078)	Locating fractions on an open number line invites the use of equipartitioning strategies based on benchmarks (e.g., halving, thirding and fifthing , etc.) and links to *fractions as number* idea. Risk: treated additively
	Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation (ACMNA079)	Potential to relate place-value system equipartitioning and *for each* idea (i.e., for each one there are 10 tenths), see base 10 system as multiplicative. Risk: introduced before students understand whole number as multiplicative system
	Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075)	Implies memorisation of facts, unclear as to how these are represented (see above). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
	Explore and describe number patterns resulting from performing multiplication (ACMNA081)	Potential to shift thinking from count of *equal groups* to *factor* or *scalar* idea to support more efficient mental strategies (e.g., 4 of anything is double double). Risk: treated as repeated addition
	No reference to arrays, regions, Cartesian product, partition or quotition division No reference to benchmark percents (50%, 25%, 10% , etc.)	
5	Identify and describe factors and multiples of whole numbers and use them to solve problems (ACMNA098)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: considered in isolation from representations of multiplication
	Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies (ACMNA100)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: strategies based on/limited to repeated addition, count all groups, rote learnt procedures
	Solve problems involving division by a one digit number, including those that result in a remainder (ACMNA101)	Highly supportive of multiplicative thinking if based on sharing or 'what do I have to multiply by' (i.e., *factor* idea)
	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)	Highly supportive of multiplicative thinking if based on equipartitioning strategies (e.g., halving, thirding or fifthing) and linked to *fraction as number* idea. Risk: treated as a iterative counting exercise
	Recognise that the place value system can be extended beyond hundredths (ACMNA104)	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *for each* idea (e.g., for each tenth there are 10 hundredths, for each hundredth there are 10 thousandths and so on)—this involves recognising recursive, exponential nature of

		the base 10 numeration system. Risk: limited to surface features
	Compare, order and represent decimals (ACMNA105)	Representing/locating decimals on a number line highly supportive of multiplicative thinking if on equipartitioning strategies (e.g., tenthing) and *for each* idea. Risk: this becomes rule based
	Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)	Potentially supportive of multiplicative thinking where numbers renamed to support more efficient calculation. Risk: this becomes rote procedure
	No link between hundredths and percentages	
6	Identify and describe properties of prime, composite, square and triangular numbers (ACMNA122)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: taught in isolation from representations
	Compare fractions with related denominators and locate and represent them on a number line	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *fraction as number* idea
	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)	Highly supportive of multiplicative thinking if related to *partition* division, *fractions as operators* and/or *think of multiplication* strategy
	Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies (ACMNA129)	Highly supportive of multiplicative thinking if based on *area* or *factor* representations of multiplication and partition division strategies (i.e., sharing and/or think of multiplication). Risk: procedures devoid of meaning, inability to check reasonableness of outcome
	Multiply and divide decimals by powers of 10	Potentially supportive of multiplicative thinking if explored in relation to structure of the base 10 system of numeration. Risk: Meaningless procedures such as ‘adding 0”, moving decimal point
	Make connections between equivalent fractions, decimals and percentages (ACMNA131)	Highly supportive of multiplicative thinking if based on equipartitioning strategies, *fraction as number i*dea—First mention of percentages. Risk: Meaningless rule-based procedures
	Investigate and calculate percentage discounts of 10%, 25% and 50% on sale items, with and without digital technologies (ACMNA132)	Supportive of multiplicative relationships if linked to equipartitioning strategies (e.g., halving, fifthing), *for each* and *fraction as operator* ideas and multiplication by decimal fractions. Risk: Meaningless rule-based procedures, inability to check reasonableness of results
7	Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)	Highly supportive of multiplicative thinking if linked to *for each* and *factor.factor.product* ideas
	Investigate and use square roots of perfect square numbers (ACMNA150)	Highly supportive of multiplicative thinking if linked to *factor* idea and *think of multiplication* strategy
	Apply the associative, commutative and distributive laws to aid mental and written computation	First mention of these properties yet used in mental computation much earlier and 2 digit by 2 digit multiplication in Year 6. Supportive of multiplicative thinking where factors used
	Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line (ACMNA152)	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *fraction as number* idea. Risk: Taught in isolation, meaningless rule-based procedures
	Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154)	Supportive of multiplicative thinking if based on equipartitioning representations, *fraction as operator*. Risk: Taught in isolation, meaningless rule-based procedures
	Express one quantity as a fraction of another, with and without the use of digital technologies	Highly supportive of multiplicative thinking if linked to *fraction as quotient* idea. Risk: Taught in isolation, meaningless rule-based procedures
	Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157)	Supportive of multiplicative thinking if linked to *fraction as quotient* idea. Risk: Taught in isolation, meaningless rule-based procedures

	Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies. (ACMNA158)	Highly supportive of multiplicative thinking if linked to *fraction as operator* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Recognise and solve problems involving simple ratios (ACMNA173)	Highly supportive of multiplicative thinking if linked to *fraction as ratio* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Investigate and calculate 'best buys', with and without digital technologies (ACMNA174)	Highly supportive of multiplicative thinking if seen as application of proportional reasoning, related to *fraction as quotient* idea
8	Use index notation with numbers to establish the index laws with positive integral indices and the zero index (ACMNA182)	Requires multiplicative thinking and recognition of f*actor* idea. Risk: laws treated in isolation from underpinning properties
	Solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies (ACMNA187)	Highly supportive of multiplicative thinking if linked to *fraction as operator* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)	Highly supportive of multiplicative thinking if linked to *fraction as ratio*. Risk: Taught in isolation, meaningless rule-based procedures
	Solve problems involving profit and loss, with and without digital technologies (ACMNA189)	This is an application of ACMNA187
	Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)	Supportive of multiplicative thinking where linked to *factor.factor.product* idea and partitioning (both additive and multiplicative)
	Factorise algebraic expressions by identifying numerical factors (ACMNA191)	Supportive of multiplicative thinking where linked to *factor.factor.product* idea
9	Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems (ACMNA208)	Requires multiplicative thinking to be achieved with understanding