Document extract

Title of chapter/article	Desperately Seeking Birthday Mates
Author(s)	D. Neal, T. Muir, K. Manuel, S. Livy & H. Chick
Copyright owner	The Australian Association of Mathematics Teachers (AAMT) Inc.
Published in	The Australian Mathematics Teacher vol. 70 no. 1
Year of publication	2014
Page range	36–40
ISBN/ISSN	0045-0685

Saturday night. Lygon Street, Carlton.
Nine mathematics teachers and teacher educators sharing a Thai banquet after a day working on a national numeracy project.
What could possibly go wrong?

desperately seeking birthday mates!

or

what maths teachers get up to on Saturday nights!

Denise Neal
Department of Education, Tasmania
<denise.neal@education.tas.gov.au>

Tracey Muir
University of Tasmania
<tracey.muir@utas.edu.au>

Kate Manuel
The Australian Association of Mathematics Teachers
<kmanuel@aamt.edu.au>

Sharyn Livy
Mathematical Association of Victoria
<slivy@mav.vic.edu.au>

Helen Chick[1]
University of Tasmania
<helen.chick@utas.edu.au>

Can you imagine a group of nine mathematics educators out for a quiet dinner in a beautiful Thai restaurant in Lygon Street, Melbourne after a busy day of work on a Saturday? Sounds like a boring event, hey?

Well, not this group! After spending a day planning for the 2013 *Reach for the Stars* event (just how much rubbish do we recycle?), the working conversation continued over dinner. We were pretty pleased with our efforts with developing ideas for 2013, and so turned our attention to possible themes for 2014. The conversation shifted to birthdays:

"Wouldn't it be a great idea if next year's *Reach for the Stars* event focussed on birthdays?"

"By the way, did you know that there only needs to be quite a small number of people in a room for there to be a good chance that there are two people with a birthday on the same day?" suggested one of the group.

"Isn't that called the Birthday Paradox?" asked someone at the other end of the table.

"How do we find out about that?" said another.

With that, it was good old Google to the rescue! A search on the phone quickly found the *Birthday Paradox*, which indicated that there only needs to be 23 people in a room for there to be a 50% chance of two people with a

1 Helen has signed an affidavit to declare that she was nowhere near Lygon Street on the night in question. The remaining authors have no alibis for the date and time of the events recounted here.

birthday on the same day, and only 57 people to have a 99% probability of a match.

What would a group of mathematics teachers do on a Saturday night but engage in some data collection and statistical investigation? We started with those at our table—"When is your birthday?"—and started accumulating dates: 29 November, 8 October, 8 May. Everyone shared, but alas no matches! Perhaps this was not surprising, as there were only nine of us, but we were not prepared to give up our quest. We were determined to find a match.

Looking about the restaurant, we estimated that there were about 60 people in the room; surely there would be a match if the information on Wikipedia and all that we knew about this problem was right! One of the group took the initiative and set about asking everyone seated in the restaurant, systematically collecting the data on a serviette—but still no matches.

We would not be defeated! A rather extroverted member of the group called over the elegant young Thai waitress and sent her to the kitchen with specific instructions to ask the entire kitchen staff for the dates of their birthdays and to write down the dates on a serviette. Dutifully she returned with all of us eagerly waiting.

Could we believe it? There were still no matches!

We still did not give up! Having paid the bill, the next step was to move our data collection activities out into Lygon Street. We met some very interesting people, including a table of men wearing brightly coloured fake fur hats that resembled bears and other unusual animals, a group of men and women smoking hookahs on the street, and lots of groups happily enjoying dinner. Each group was accosted by our 'leader', who introduced us with these words: "We are maths teachers and we are conducting an important survey. We need to know when your birthday is as we are trying to find two birthdays that match." Amazingly, no-one said, "No". They actually took us quite seriously and told us their birthdays, which we added to the ever-growing list of dates on our serviette. We thought that there might be a skewed sample, in that it was Saturday night (4 May), and we logically expected that there would be a few people out celebrating birthdays, but this did not seem to be the case. There was much laughter as we continued our quest!

Finally we had a match: two birthdays on 17 May! It had taken much longer than expected and many more than 57 people. Much laughter and fun was had by all, proving that maths can be fun, that statistics can be collected in many places and used for a range of purposes, that sometimes reality can go against what we think we should expect from the theoretical probability, and that really rare events do happen.

The maths behind the birthday problem

The best (only?) way to determine the likelihood of a birthday match within a group of people actually involves thinking about the opposite (complementary) problem: what are the chances of having *no* matches in the group? If we can find out the probability of having no matches at all in a group, then the probability of finding a match will be 1 minus this probability, since 'having one or more matches in the group' is the complementary event for 'having no matches at all'.

To find out the probability of having a group with no matches, we will imagine trying to create a room full of people, no two of whom share a birthday. To begin with, imagine a big room with just one person in it. In rather unimaginative fashion we will call this individual Person-1. With Person-1 being the only person in the room, it is obvious there are no 'twin' birthdays in the room.

Now suppose that a new person, Person-2, comes to the door. Person-2 is only allowed into the room if he or she does not share a birthday with Person-1. What are the chances of that happening? Ignoring 29 February (to keep things simple), the chances are

$$\frac{364}{365}$$

as Person-1 has used up one of the birthdays in the year. This means that the chance of having two people in the room who *do not* share a birthday is

$$\frac{364}{365}.$$

We can now deduce that when there are two people the chance of *having* a match—which is the 'opposite' or 'complement' of not having a match—is the complement of the 'no matches' probability, or

$$1-\frac{364}{365}.$$

Not surprisingly, for two people this turns out to be

$$\frac{1}{365}.$$

Things will, however, get more complicated from now on!

Let us keep going. Imagine that Person-3 turns up. This person is only allowed into the 'no matches' room if his or her birthday does not match either of the two people already in the room. Since those two people have used up two birthdays from the year, the chance of being able to let Person-3 into the room must be

$$\frac{363}{365}.$$

From this we can conclude that the chances of having three people in the room *without* a common birthday must be

$$\frac{364}{365}\times\frac{363}{365}$$

since we need to have a non-matching second person *and then* a third person who does not match either of Person-1 or Person-2. The complement of this,

$$1-\left(\frac{364}{365}\times\frac{363}{365}\right)$$

then gives the probability of the three people in the room actually having a matching pair of birthdays (or even more than one), since

$$\text{P(at least one match)} = 1 - \text{P(no matches)}.$$

If Person-4 turns up, the probability that he or she will be allowed into the 'twin-free' room is

$$\frac{362}{365}$$

and so the chances of having a four-person 'no matches' room is

$$\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}.$$

This implies that the chances of *having* at least one birthday match with four people is

$$1-\left(\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\right).$$

If we continue our argument, allowing people into the room *only if* they *do not* share a birthday with anyone already in there, then you can see that it becomes progressively less likely that a person will be allowed to enter because there are more and more birthdays already in the room and any new arrival cannot match up with any of these. Thus, the chance that Person-k, the kth person to arrive, will be allowed into the room is

$$\frac{365-k+1}{365}.$$

The resulting probability for being able to make a room full of k people, none of whom share a birthday, is thus

$$\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\ldots\times\frac{365-k+2}{365}\times\frac{365-k+1}{365}$$

(there are $k-1$ numbers in that big long product).

Our complement argument says that the event of having at least one match is the complement of having no matches at all, and so

$$\text{P(at least one match)} = 1 - \text{P(no matches)},$$

and so we can now conclude—*drum roll*—that the chances of *having* one or more birthday matches in a room of k people is

$$1-\left(\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\ldots\times\frac{365-k+2}{365}\times\frac{365-k+1}{365}\right).$$

We can set up a spreadsheet to work out the values of this probability for different values of k. The results are shown in Figure 1. A graph of these probabilities as a function of group size is shown in Figure 2. As can be seen, 23 people are needed in order for the probability of having a shared birthday in the group to be at least 50%, but it then continues to climb. Once there are 40 people in the group, the chance of a shared birthday is

A	B	C	D	E
Number of people	Number of available days	Col B value/ 365	Cumulative product of entries in C = P(no matches)	Chance of birthday matches (= 1 - Col D value)
1	365	1.0000	1.0000	0%
2	364	0.9973	0.9973	0.27%
3	363	0.9945	0.9918	0.82%
4	362	0.9918	0.9836	1.64%
5	361	0.9890	0.9729	2.71%
6	360	0.9863	0.9595	4.05%
7	359	0.9836	0.9438	5.62%
8	358	0.9808	0.9257	7.43%
9	357	0.9781	0.9054	9.46%
10	356	0.9753	0.8831	11.69%
11	355	0.9726	0.8589	14.11%
12	354	0.9699	0.8330	16.70%
13	353	0.9671	0.8056	19.44%
14	352	0.9644	0.7769	22.31%
15	351	0.9616	0.7471	25.29%
16	350	0.9589	0.7164	28.36%
17	349	0.9562	0.6850	31.50%
18	348	0.9534	0.6531	34.69%
19	347	0.9507	0.6209	37.91%
20	346	0.9479	0.5886	41.14%
21	345	0.9452	0.5563	44.37%
22	344	0.9425	0.5243	47.57%
23	343	0.9397	0.4927	50.73%
24	342	0.9370	0.4617	53.83%
25	341	0.9342	0.4313	56.87%
26	340	0.9315	0.4018	59.82%
27	339	0.9288	0.3731	62.69%
28	338	0.9260	0.3455	65.45%
29	337	0.9233	0.3190	68.10%
30	336	0.9205	0.2937	70.63%
34	332	0.9096	0.2047	79.53%
35	331	0.9068	0.1856	81.44%
36	330	0.9041	0.1678	83.22%
37	329	0.9014	0.1513	84.87%
38	328	0.8986	0.1359	86.41%
39	327	0.8959	0.1218	87.82%
40	326	0.8932	0.1088	89.12%
41	325	0.8904	0.0968	90.32%
48	[illegible]	0.8712	0.0394	96.06%
49	317	0.8685	0.0342	96.58%
50	316	0.8658	0.0296	97.04%
51	315	0.8630	0.0256	97.44%
52	[illegible]	[illegible]	0.0220	97.80%
57	309	0.8466	0.0099	99.01%
58	308	0.8438	0.0083	99.17%
59	307	0.8411	0.0070	99.30%
60	306	0.8384	0.0059	99.41%
61	305	0.8356	0.0049	99.51%
69	297	0.8137	0.0010	99.90%
70	296	0.8110	0.0008	99.92%
71	295	[illegible]	0.0007	99.93%
79	287	0.7863	0.0001	99.99%
80	286	0.7836	0.0001	99.99%
81	285	0.7808	0.0001	99.99%
88	278	0.7616	0.0000	100.00%
89	277	0.7589	0.0000	100.00%
90	276	0.7562	0.0000	100.00%
91	275	0.7534	0.0000	100.00%
92	274	0.7507	0.0000	[illegible]

Figure 1. The probability that there will be at least one shared birthday in a group for different sized groups of people.

almost 90%. With 83 people we are very close to 100%. If our mad Lygon Street adventurers had to ask 70 people (or even more—they lost count!) before getting a match, then they experienced a very rare event indeed: there is a greater than 99% chance of getting a match with 69 people, and, in fact, the chance of not getting a match with so many people is only about 0.1% or 0.001. Of course, once you have 366 people (or 367 if you want to allow for 29 February birthdays) a match is guaranteed, but, until then, there is a small (possibly very small) chance that you will have a twin-free group—it just could be very *unlikely* (unless you happen to be mad maths diners on Lygon Street)!

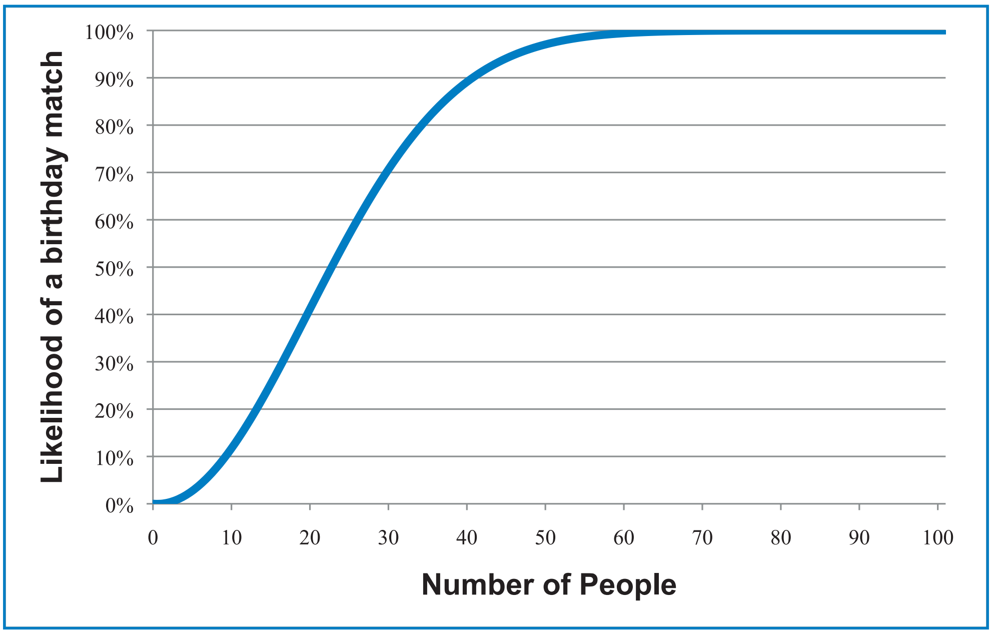

Figure 2. Graph showing the likelihood of a birthday match
for a given number of people in the group.