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CHAPTER 8.

ASSESSING STUDENTS' UNDERSTANDING OF FRACTION EQUIVALENCE*

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Learning is a process that builds upon students' prior knowledge. Children's existing understandings guide their interpretation, understanding and incorporation of new information (National Research Council, 2001a). Teachers need to understand and investigate the mathematical thinking that students utilise to solve fraction problems, in order that they may advance students' knowledge and understanding. Many teachers frequently observe students in their classrooms use inappropriate whole number strategies when solving fraction problems. When students exhibit these errors, they provide opportunities for teachers to adapt their lessons to address such errors and guide students' mathematical thinking towards improved understanding.

We can identify students' misconceptions by posing tasks that provide insight into their thinking. This chapter focuses on the assessment of students' knowledge and conceptual understanding of equivalent fractions using tasks from a pencil and paper assessment instrument. The tasks were developed as part of a project involving over two years of research and 640 students from Years 3 to 6 (approximately 8 to 12 years of age). First, we describe what conceptual understanding of equivalent fractions encompasses. This is followed by the presentation of four tasks that teachers can use to assess students' knowledge. The tasks require students to represent equivalent fractions using area models and to construct symbolic equivalents. They incorporate both "skill" questions that require

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the application of a practised routine or procedure, and "conceptual" questions that require students to apply their knowledge and explain their actions. Included with these tasks are the common errors exhibited by students, accompanied by examples of students' work or excerpts of their explanations from interviews.

Understanding fraction equivalence

Ni (2001) contends that in many mathematics classrooms, understanding of fraction equivalence is considered mastery of the rule, "multiply or divide the numerator and denominator of a fraction by the same number" (p. 413). Research shows that it is much more (e.g., Lamon, 2005). Understanding fraction equivalence necessitates students recognise that two or more fractions can represent the same quantity, thus belonging to an equivalence set. For example, the equivalence set for the fraction $\frac{3}{4}$ can be represented symbolically as

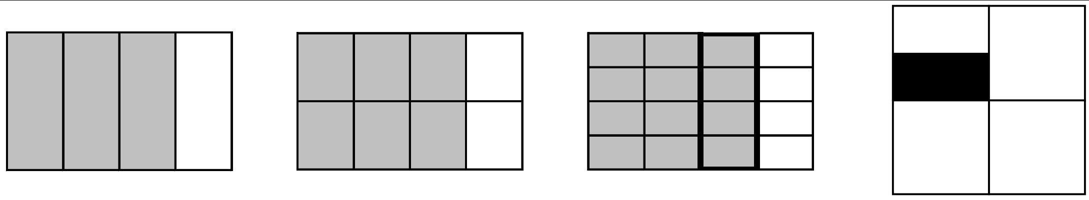
$$\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\right\}.$$

Each fraction within the set is interchangeable with another as they refer to the same relative amount—three-quarters (Cathcart, Pothier, Vance, & Bezuk, 2006). This relationship can be represented using manipulatives, spoken language, written language including symbolic notation, real life scenarios and pictorial images (Lesh, Landau, & Hamilton, 1983) including area, number line, collection and segment models (Cathcart et al., 2006; Niemi, 1996). Students who possess conceptual understanding of fraction equivalence can seamlessly link and manipulate differing representations. They "see the connections among concepts and procedures and can give arguments to explain why some facts are consequences of others" (National Research Council, 2001b, p. 119).

Pictorial representations

Pictorial representations of part/whole area models can be described as "simple representations" when the total number of equal parts in the representation matches the fraction denominator. The shaded part is associated with the numerator and the entire representation is associated with the denominator. For example, the fraction three-quarters is shown in Figure 8.1(a). Equivalent pictorial representations occur when the number of equal parts of the whole is a multiplicative factor less than or greater than the denominator (Niemi, 1996) as shown in Figures 8.1(b) and 8.1(c).

The area of the whole and shaded part never changes, but the number of equal parts into which the whole is divided can alter. Thus different fraction names can be offered for the shaded area and elements within an equivalence set identified.



(a) original (b) partitioned (c) partitioned again (d) unequal parts

Figure 8.1. Pictorial representations for three-quarters and an unconventional representation for one-eighth.

Alternatively, if the representation of twelve-sixteenths is provided, the process of "chunking" by constructing larger parts could be applied to help solve tasks involving equivalence (Lamon, 2005). Chunking allows the whole to be subdivided into equal-sized parts with no remainders. If the chunk was a column, as shown in Figure 8.1(c), then the horizontal lines could be ignored and the fraction three-quarters recognised. In all instances, the referent unit or whole *does not alter*. Additionally, students must recognise that naming a fraction requires the division of the whole into equal-sized parts. Errors can arise in naming the correct fraction if the significance of equal-sized parts is not recognised. For instance, students frequently suggest the fraction shaded in Figure 8.1(d) represents one-fifth.

Thinking exhibited

Representations for "a whole"

In this section, four equivalent fraction tasks are presented and discussed in terms of the typical strategies students employ to complete each type of task. The first two tasks (see Figure 8.2) examine students' knowledge of one or a whole. Task 1 is a "skill" question and can be answered by the application of a practised routine or procedure. Task 2 represents a "conceptual" question that requires students to apply their knowledge and explain their actions.

Circle the fractions that are **equal** to 1.	Shade in $\frac{2}{2}$ of the shape below?
$\frac{8}{8}$ $1\frac{1}{100}$ $\frac{1}{1}$ $\frac{9}{10}$ $\frac{4}{4}$ $1\frac{1}{8}$ $\frac{9}{8}$ $\frac{7}{8}$ $\frac{10}{9}$ How did you work this out?	

(a) Task 1 — procedural

(b) Task 2 — conceptual

Figure 8.2. Equivalence tasks examining the notion of one or a whole.

Students' strategies for solving Task 1 can be obtained by asking the question, "How did you work it out?" Incorrect responses often result when students consider the individual digits in a fraction rather than considering the value of the fraction in its entirety. In our study, a Year 3 student wrote, "I got all the numbers that had 1 in it", whilst another Year 3 student explained, "If it has 1 on the side or on the top it is equal to 1". These students applied flawed procedural knowledge and exhibited limited understanding.

Correct strategies employed by students are listed in Figure 8.3. The procedural strategy to compare the numerator and denominator was observed, but students considered the two numbers together. Some students linked their knowledge from other domains to improve their understanding of fractions (e.g., see Sam's response in Figure 8.3). In contrast, other students attempted to develop further understanding by creating an appropriate image (e.g., Julie). Indeed, correct strategies that students employ offer teachers instructional ideas for rectifying other students' misunderstandings.

Strategy and examples of students' explanations
Compare numerator and denominator:
"Well I worked it out by looking if the two numbers are the same like $\frac{1}{1}$" (Max)
"The numbers with the same number top and bottom are wholes" (Abbie)
"I looked at the numerator and denominator and checked if both numerator and denominator have the same numbers" (Kristen)
Linking to other related mathematical domains:
"Because everyone circled is 100% which is equal to 1" (Sam)
Creating an image:
"I pictured a circle in my mind and thought if $\frac{8}{8}$ is the hole [sic] circle coloured than it must be a hole [sic]" (Julie)

Figure 8.3. Correct strategies employed to identify all fractions equal to 1.

The stability of students' knowledge and an indicator of their understanding of the equivalence set one $\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots\right\}$, was inferred from a review of responses for both tasks. Many students were able to identify symbolic fractions equal to one, but, were unable to apply their knowledge to equivalent pictorial representations. The most common error was to shade two of the parts or half of the shape provided in Task 2. A Year 4 student wrote, "I looked at the numerator and denominator to see if it is the same." Yet she was unable to represent $\frac{2}{2}$. She stated in the interview: "I found this hard, because this [pointing to $\frac{2}{2}$] is actually one whole [motions hand across the length of the entire shape shown in Task 2], but it has four pieces in it…". She shaded two of the partitions and recognised it as one half, yet she was unable to reconcile the discrepancy in her symbolic knowledge of fractions and pictorial images. She eventually offered the reason, "Because there's only four pieces and if the numerator says two and the denominator says two, so that's why I had it as two."

It is important to ascertain students' thinking as a correct answer does not always reflect correct reasoning. A Year 5 student correctly selected all the fractions equal to 1, "because whole numbers equal 1". However, he shaded only half of the shape in Task 2. During the interview, he was asked whether shading two of the four parts was correct. Although he was unsure, he said it was wrong and then incorrectly reasoned: "And it says two twos, two over two. Oh. Now I get the answer! 'Cause two [points to the numerator and two parts] over two [points to the denominator and the remaining two parts]. One whole. So cover all of them." Hence, not only is it important to examine students' responses to procedural and conceptual written tasks, it is equally important to ascertain their thinking as flawed reasoning can result in the correct answer.

Applying understanding of fraction equivalence

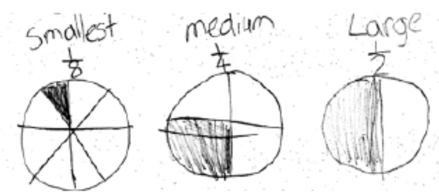
Task 3 and Task 4, (see Figure 8.4) are generally suitable for students in Years 3 and 4 or Years 5 and 6 respectively. For each task, students are required to apply their understanding of fraction equivalence and integrate suitable diagrams to explain their reasoning.

Using pictures and words, explain how you would work out which of these fractions is smallest. $\frac{1}{8}$ $\frac{1}{2}$ $\frac{1}{4}$	Using pictures and words, explain how you would work out which of these fractions is smallest. $\frac{2}{6}$ $\frac{1}{2}$ $\frac{2}{3}$
(a) Task 3 for Years 3 and 4	(b) Task 4 for Years 5 and 6

Figure 8.4. Questions requiring students to apply their understanding of fraction equivalence.

A review of responses to Task 3 by students in our study is shown in Figure 8.5.

<table>
<tr><th>Correct</th><th>Incorrect</th></tr>
<tr><td>Consistent-sized referent unit



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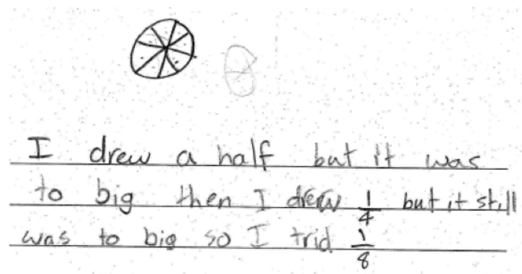
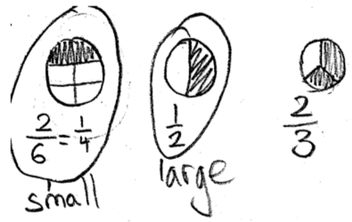
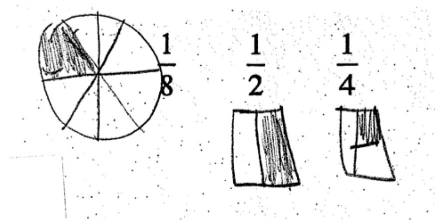


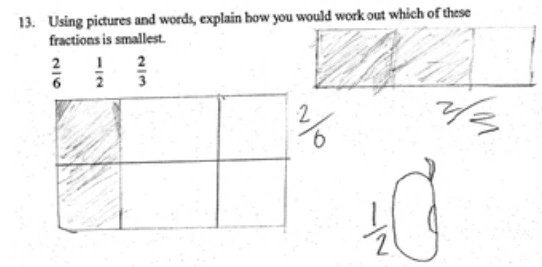
</td><td>Incorrect partitioning



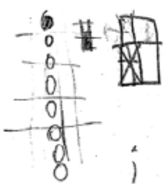
Different-sized referent units

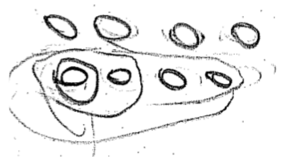


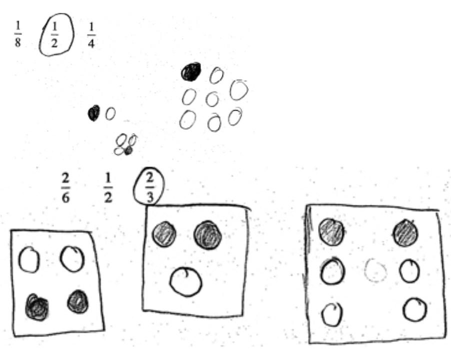
and



Mixing fraction models


</td></tr>
<tr><td>Appropriate use of collection model


</td><td>Inappropriate use of collection models



</td></tr>
</table>

Figure 8.5. Pictorial representations employed to identify the smallest fraction.

Overall, responses highlight the importance of starting with an equally divided referent unit. In particular, incorrect responses show students' lack of understanding of the importance of a standard referent unit when comparing fraction quantities. For Task 3, the appropriate referent unit is eighths, since both $\frac{1}{2}$ and $\frac{1}{4}$ can be conveniently converted to eighths. Different-sized referent units, a mixture of collection and area models, and inappropriate use of collection models were among the errors made by students. From an instructional perspective, a common referent unit is a fundamental concept necessary for understanding fraction equivalence across all grades.

Although Tasks 3 and 4 requested students use pictures to explain their answers, they were absent from many responses. Strategies that were considered correct in the absence of pictures are listed in Figure 8.6 and were similar to those found by other researchers (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Smith, 1995). Explanations used to compare the size of fractions incorporated "a quantitative notion, or awareness of the 'bigness' of fractions" (Bezuk & Bieck, 1993, p. 127). The development of quantitative understanding of fractions allows students to: (a) judge the relative size of fractions and in relation to a single reference point, whether it is the fraction one-half or another fraction, and (b) view fractions as a smaller part of a unit or a measure of a quantity which has been divided into smaller parts. In contrast, the mathematical language for other strategies used by students—checking the numerator and/or denominators—varied. Most explanations incorporated the notion of "the bigger the denominator, the smaller the parts". It is also possible that these two strategies relied more on the use of procedural knowledge and that students do not possess enough understanding to incorporate pictorial representations.

Strategy and examples of student explanations
Comparing size: "One eighths is two quarters and if one half is two quarters" "One eighth because two and one eighth is one quarter and two and one quarter is one half" "Well one half is pretty big and one quarter is half of that and one eighth is half of that" Check numerators and denominators: "If the numerators are the same, then the largest denominator is the smallest fraction. Therefore one eighth is the smallest." "First you look how small the numbers are on the top and find the biggest number on the bottom" "On the numerator it is one. But on the denominator is different. The more bigger denominator, the smaller. But if the numerator is like four. Then it could be bigger." Check denominators: "I know because the highest number on the bottom means more smaller pieces" "By finding the highest number being cut into smaller blocks"

Figure 8.6. Written explanations for identifying the smallest fraction correctly without accompanying diagrams.

Pictorial and symbolic strategies employed by students from Year 5 and 6 to select the smallest fraction from those listed in Task 4 included those strategies used by Year 3 and 4 students in answering Task 3. However, written explanations also incorporated the common denominator strategy: "Change all the denominators to six and make $\frac{1}{2}$ into $\frac{3}{6}$, $\frac{2}{3}$ into $\frac{4}{6}$ and keep $\frac{2}{6}$ the same" (Year 6 student). This strategy was absent from Year 3 and 4 student responses, most likely due to Task 3 incorporating unit fractions (i.e., fractions with a numerator of 1). A reference point strategy was employed by Year 5 and 6 students in which $\frac{1}{2}$ was used as a reference point and the remaining two fractions were changed to a common denominator. A Year 6 student, for example wrote, "Well one half is one half and two thirds are two thirds but $\frac{2}{6}$ is equivalent to one third and one third is smaller than one half and $\frac{2}{3}$". Other students used half as a reference point without changing the remaining fractions, possibly showing a greater understanding of the size of the fractions $\frac{2}{6}$ and $\frac{2}{3}$. For example, a Year 5 student wrote, "2 over 6 is the smallest, because half is more, and two thirds is more than half". These students exhibit a quantitative understanding of fractions that is crucial for evaluating the reasonableness of fraction computations (Bezuk & Bieck, 1993).

Incorrect written explanations are listed in Figure 8.7. All explanations exhibit some form of whole number reasoning in which the fraction numerals are decomposed into separate numbers and mathematical comparisons are applied to the individual digits.

Strategy and examples of student explanations
Whole number reasoning: Considering both numerator and denominator "$\frac{1}{2}$ are smallest because they are lower numbers" "Because they all have one up the top so it goes like this $\frac{111}{248}$" Comparing denominators "$\frac{1}{2}$ I know because 2 is smaller than 4 and 8 is bigger than 4" Adding numerator and denominator "$\frac{1}{2}$ because all they do is add up to 3" Incomplete reasoning: "$\frac{1}{2}$ even though it is a small number it takes up the most space than $\frac{1}{8}$ and $\frac{1}{4}$."

Figure 8.7. Written explanations of strategies resulting in an incorrect answer.

Overall, we found that students used procedural knowledge when answering equivalent fraction problems presented in symbolic form. In some instances, whole number reasoning was demonstrated in the procedures they used. Many students were unable to represent a symbolic fraction using an equivalent area diagram. Students who successfully linked symbolic and pictorial part/whole area interpretations for one whole demonstrated their deeper understanding by applying their knowledge to equivalent area models. Furthermore, students who were able to answer either Task 3 or Task 4 correctly were able to select the appropriate pictorial representation to illustrate their reasoning. Importantly, these students understood the need for a common referent unit. Students who did not incorporate a pictorial representation in their explanation, but exhibited a quantitative understanding of fractions, demonstrated that they possessed conceptual understanding of fraction equivalence.

Implications for teaching

An examination of students' responses to the tasks presented here, show that students who incorporate:

(a) the notion that a fraction represents a quantity;
(b) mathematical terminology in a correct context;
(c) comparison of fraction quantities;
(d) meaningful use of pictures; and
(e) comprehensive explanations;

exhibit greater levels of conceptual understanding of fraction equivalence. Consequently, students who possess these essential understandings are able to apply their knowledge and explain their actions demonstrating strong links between a range of skills and knowledge, and mathematical representations of fractions.

Students' conceptual understanding of fraction equivalence or any other mathematical concept needs to be ascertained from a variety of carefully considered tasks. The four pencil and paper tasks described in this chapter have been shown to be effective in identifying students with limited and/or incomplete knowledge of fractions. Notwithstanding, it is equally important for teachers to ask students to explain their thinking during regular classroom activities to confirm their understanding (Siegler et al., 2010). Carefully chosen tasks, combined with good questioning, can reveal useful information about students' thinking and strategies that can be used by teachers to plan learning experiences to consolidate correct reasons, expand their repertoire of strategies or correct misconceptions and incomplete reasoning.

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